Notes towards a qualitative analysis
of information in measurement results

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Abstract: the work analyzes the various components of information brought by measurement results, in particular highlighting the meaning and the role of semantic and pragmatic information present in measurement. It is suggested that several activities of the measurer, e.g., calibration, are based on more-than-purely-syntactic information: a clear identification and formalization of semantics and pragmatics of measurement is therefore required to achieve actual intelligent measurement systems.

Keywords: information in measurement; information quality and information quantity; intelligent measurement systems

1. Introduction

Measurement is an operation aimed at *acquiring information* on the measured thing with respect to an attribute. A clear identification of the information quality and quantity conveyed by a measurement result is therefore the ultimate condition to establish whether a measurement is worth to be performed. This decision is traditionally made by human beings, usually measurers themselves, who evaluate the trade-off costs/benefits, i.e., estimated resources required to perform the measurement vs. estimated information obtained by measurement results. In this way, they judge whether to measure or not, usually on the basis of informal criteria such as their experience and their acquaintance with the available measurement systems.

The progressive introduction of intelligent measurement systems [1] is a turning point in this regard. A peculiar feature of such systems is indeed their ability to implement different strategies to accomplish the measurement, and to automatically select one of them as the best

one in the given context and for the given goals [2]. Such a selection should obviously take into account the above mentioned trade-off costs/benefits of the measurement: an important step towards the achievement of actual intelligent measurement system is then the formal characterization of the information brought by measurement results.

A formalized and quantified concept of information is well known, due to the work of researchers such as H. Nyquist, R. Hartley, and particularly C. Shannon, whose theory arose and found its most important interpretation in communication systems. Such a concept of information, and those defined from it (entropy, equivocation, channel capacity, …), is syntactic in nature, i.e., it only refers to symbols and their combination rules, with no reference to any possible "meaning" or "utility" brought by them. In a common model, the act of selecting a symbol from a defined set is assumed to convey a given information, whose quantity is related to the a priori uncertainty recognized on that specific selection: the more a priori uncertain was the symbol, the more is the information quantity obtained in its selection. Likewise but in subjective terms, the more the observer is surprised in seeing that the symbol has been selected, the more is the information quantity he achieves.

As typical in science, this simplification allowed us to reach some very powerful quantitative results (in particular those concerning entropy as a lower limit in information compression and channel capacity as an upper limit in information transmission), at the price of the renouncement to model a more general, and maybe sometimes more adequate, concept of information.

The applicability of this purely syntactic notion of information in metrology is the subject of several publications (e.g., [3] and [4]; the author's position in this regard is presented in [5]). The fundamental issue to consider for this application is the conceptual difference in the intended task of transmission systems (TS) and measurement systems (MS), and therefore in the way their quality is evaluated: while a TS is aimed at *transferring* information, the goal of a MS is *to capture* information, in principle unknown before measurement itself.

More recent and less explored is the *general meaning* of the information conveyed by measurement results, including thus some semantic and pragmatic components. Some works have already pointed out the interest for such a topic (e.g., [6] and [7]), and in particular [8] identifies the "relationships between energy, information and meaning" among the basic unsolved problems in the current measurement science, requiring "the development of models of the meaningful aspect of measurement".

Semantic information has already been the subject of several studies, done in particular during '50s, '60s, and '70s by logics and philosophers of science such as Y. Bar-Hillel, R. Carnap, and J. Hintikka (cf., e.g., [9] and [10]). The attempt was done to conceptualize and formalize the semantic information conveyed by a proposition as the *content* of the proposition itself. With such a broad connotation, the problem remained largely unsolved and in the following years was substantially left aside. Our claim is that in the specific case of measurement (i.e., when the issue is to characterize semantic information of that particular kind of propositions expressing the fact that a given value is assigned to a given thing for a given attribute) a satisfactory solution can be reached: the present paper aims at analyzing some foundational issues in this view. In particular, it is asserted here that the information brought by measurement results *has several components*, the syntactic information formalized in Shannon theory being just one of them. Such components, depending on each other although conceptually well defined and distinct, will be called *information-from-selection*, *information-from-structure*, and *information-from-connection*.

As it will be discussed, these different *qualities of information* of measurement results have complementary roles: while the information-from-selection coincides with the syntactic component of the information, the information-from-structure and the information-from-connection are peculiar to measurement. It will be hypothesized that the information-from-structure relates to the semantic level of the measurement results, i.e., how the symbols expressing such results can be interpreted, and that the information-from-connection pertains to the pragmatic level, i.e., how the symbols expressing the measurement results can be used.

2. Information-from-selection

Measurement can be thought of as an operation leading to the selection of a symbol from the set of all measurement results that are a priori considered as possible. As such, it can be interpreted in the context of the Shannon's information theory. The selected symbol conveys information, it will be called *information-from-selection*, since its knowledge reduces the uncertainty that was present before the selection: a different symbol could have been chosen, but the selected one has been indeed selected. The degree of uncertainty reduction, and therefore the quantity of information-from-selection, depends on the number of symbols that can be selected a priori, and, in the more general case, on the probability of the symbol selection: the less the probability $p(x_i)$ that a symbol x_i is selected, the more the information quantity $I(x_i)$ obtained by its selection. As Shannon proposed, $I(x_i)=\log_2(p(x_i))$ bit.

Both TSs and MSs act as symbol selectors: given a set of a priori possible values, one of these is selected as the result of the transmission / measurement. If the system resolution is enhanced (then the cardinality of the possible results is increased), then the obtained results usually convey a greater quantity of information-from-selection. In the metrological interpretation of the information-from-selection some peculiarities are however present that distinguish it from the syntactic information proper of the communicational model.

A TS can be formalized as an entity able to take symbols $x_i \in X$, generated by some external source entities, and as a consequence to produce symbols $y_j \in Y$, its "readings", at its output. A TS operates in ideal conditions whenever for each input symbol x_i a given output symbol y_j is determined. Since this does not generally happen because of several "error" causes, the relation between TS input and output is only uncertainly known, for example in statistical terms by means of the conditional probabilities $p(y_j|x_i)$. It is assumed that the TS input and output symbols are not simultaneously known, although in principle both knowable. From the knowledge of the reading symbol y_j (and of the characteristics of the TS and the source) the corresponding input symbol has then to be inferred by means of a suitable computation.

On the other hand, a MS can be formalized as an entity able to interact with some external entities with respect to a given measurand and as a consequence to produce symbols $y_j \in Y$, its "readings", at its output. Each measurand is characterized by the range set X of such a mapping, whose elements represent the possible values that can be assigned to measured things with respect to that measurand. From the knowledge of the reading symbol y_j (and of the characteristics of the MS and the measured thing) a value x_i has to be inferred by means of a suitable computation, assumed as representing the measurand value. x_i can be a single element of X or, in the more general case (e.g., to emphasize the presence of some intrinsic uncertainty on the measurand definition), a collection of elements of X, i.e., a subset, possibly "graduated" by a probability or a possibility distribution (in this case x_i becomes a fuzzy subset), of elements of X.

Therefore, both TSs and MSs produce as their output a symbol from which their input has to be inferred. But while a "true" (and independent of the transmission) value for the TS input variable is assumed, the hypothesis of existence, or at least of knowledge independent of measurement, of a true value for a measurand does not seem to be maintainable. TS and MS have thus different goals, and their quality is judged according to different criteria, as the so-called "universal sensing system model" (presented in [8] and further analyzed in [11]) makes clear.

The information-from-selection conveyed by a measurement result is affected by the specificity of the value assigned to the measurand. As discussed in [5], in the case in which the measurand value x_i is formalized as (crisp) subset of X (corresponding to an "expanded uncertainty", as defined in [12]), the quantity of information-from-selection can be computed as $I(x_i)=\log_2(\#X/\#x_i)$ bit (where the operator $\#$ denotes the cardinality of the set), so that $I(x_i)$ decreases as the specificity of x_i, here formalized by $\#x_i$, decreases. In the case of complete specificity, when the subset x_i reduces to a singleton, $\#x_i=1$, this becomes $I(x_i)=\log_2(\#X)$ bit, i.e., the usual Hartley-Shannon definition. In the opposite situation, when the measurement result

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simply reports the whole range of a priori possible measurand values, $\#x_i = \#X$, the definition leads to $I(x_i)=0$ bit: in such a case the measurement is indeed completely uninformative.

In such a definition the dependency on the measurand range set X should be noted: it is indeed inherent to the nature of a selection to convey information that is *conditional* to the set on which the selection itself is performed. The value of the information-from-selection conveyed by a measurement result can be thus maximized by a suitable choice of the set X, a fact that highlights the conventional basis of the concept.

The information-from-selection is just the iceberg tip of the knowledge involved in and conveyed by measurement: behind symbols there are meanings, and these are crucial to comprehend the non purely syntactic nature of measurement.

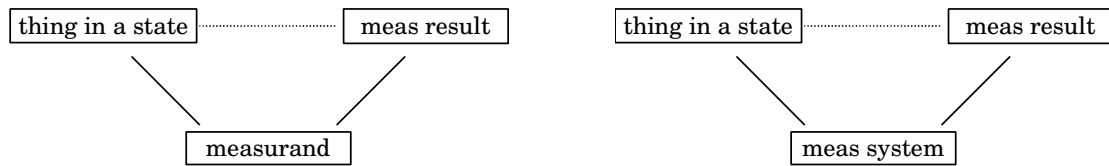## 3. Bridge: at the basis of a semantic information in measurement

Measurement establishes a relation between states of things (i.e., specific time versions of things) and symbols with respect to attributes, measurands being thought of here as names for ways to map things under measurement to symbols [13]: this relation lays the foundation for semantics in measurement. Indeed, basic semantic concepts are usually modeled by means of the *semiotic* (or *meaning*) *triangle*:
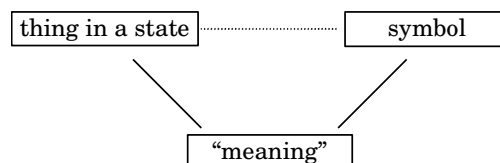
expressing that meanings are peculiar mediators in the relation that holds between symbols and what is designated by them, i.e., things in given states [14].

In the specific case of measurement the semiotic triangle admits two complementary interpretations: meanings are here (or: are maintained here by) both measurands and measurement systems. In the former, "theoretical", acceptation, measurement results have a meaning (i.e., convey some semantic information) being measurand values; in the latter,

"operational", acceptation, measurement results have a meaning being the results of an operation performed by a given system:

| thing in a state | ············ | meas result | | thing in a state | ············ | meas result |
| measurand | | meas system |

In both cases, semantics emerges as inherent to measurement.

Although from quite different premises, [6] reaches the same conclusion, highlighting that this meaning relation is precisely the subject of the representational approach to a theory of measurement, that indeed formalizes measurement as that specific kind of functional relation that is the homomorphism [15].

Such a relation is ideally based on a (commonly left implicit) axiom of stability of the reference (what the philosophy of language would call the *rigidity* of the reference): if the thing state does not change then also the symbol associated with the thing must not change. This guarantees that measurement results are faithful substitutes for measured things with respect to the measurand [16], [17]). The confirmation of actual stability is an operational, and not theoretical, issue, involving the check of the repeatability and reproducibility of the measurement system. In the case two consecutive measurements produce different results, the *cause* can be recognized in a non-ideal behavior of the measurement system or in a change of the thing state: this ambiguity can be solved only in reference to a model in which both the measured thing and the measurement system are described. This is clearly *a semantic issue*. Moreover, this is just an instance of the "calibration problem", which is central in both foundations of measurement and operative metrology: from the instrument reading value generated by the measurement system the actual measurement result, therefore a measurand value with a given associated uncertainty, has to be inferred [18], in particular specifying the number of its significant (*significant …*) digits. It can be noted that, in its turn, this is just an instance of an even more general problem that [19] states as the specification of a suitable *observation statement* from a given *observation report*.

What we are claiming here is therefore that *calibration is a semantic-based activity*. An intelligent measurement system can be empowered to perform tasks such as its self-calibration only if some semantics is embodied in its knowledge base: semantics thus plays here the role of a broad context in which syntactic issues can be interpreted and answered. Such a context is worth of further analysis.

4. Information-from-structure

Measurands are always evaluated on the measured thing *relatively to a reference*, so that the expression of measurement results requires the explicit indication of such a reference. The same symbol "3" assumes different meanings if assumed relatively to different measurement references, "meters", or "nanometers", or "Kelvin degrees", or "Richter scale degree". The measurement reference (usually, but not necessarily, the measurement unit) *establishes a meaning* for the symbol representing the measurand value.

As a well known fact, measurement references can be formalized in terms of *measurement scales*, each of them characterized by a given *type*. The most commonly considered scale types (nominal, ordinal, interval, ratio, absolute) can be linearly ordered according to the algebraic structure they imply on the set of symbols adopted to express measurand values [20]. As an example, the ordinal type enriches with an order relation the nominal type, only identified by an equivalence relation.

Measurement and scale definition are different operations, and any given measurement requires the measurand scale to be previously defined. Measurand values are embedded in the algebraic structure specific of the scale type in which the measurand is evaluated. As such, they inherit the information conveyed by their specific scale type: this peculiar information component will be called *information-from-structure*.

This kind of information is somehow analogous to what [21] defines "nested information": from the knowledge that the sum of the internal angles of all triangles is π rad and that a given geometrical figure is found to be a triangle one is able to deduce that such a figure will have in

its turn π rad as the sum of its internal angles (from a logical point of view this is merely a syllogism). Because of the first discovery, the recognition that a specific figure is a triangle conveys also such an inherited information. Correspondingly, the scale type of a measurand defines the context for the measurement result and affects the possibility of its subsequent formal treatment. The case of nominal scale measurements is the weakest, since all that can be inferred from the knowledge of the measurement results is whether two measured things are equivalent with each other with respect to the measurand (i.e., they produce the same measurement result) or not. Opposite is the situation of measurements performed in algebraically richer scales, such as absolute and ratio scale, in which the information-from-structure brought by measurement results is much higher, allowing to assess not only whether two things are equivalent to each other, but also, e.g., whether with respect to the measurand one thing "doubles" a second thing (i.e., two suitably combined replicas of the first thing are equivalent to the second thing).

The degree of information-from-structure conveyed by a measurement result depends thus on "how much the measurand scale is structured": the richer the structure of such a scale, the higher the information-from-structure degree. In more formal terms, the more specific the class of admissible transformations for the scale [22], the more the information-from-structure conveyed by the measurement results.

In the comparison of two measurement results x_1 and x_2, the information-from-structure degree is an indicator of the possibility of assigning an informative content to the question: "*how much x_1 and x_2 are equal to each other?*". In nominal scale measurement the answer can only be boolean: the two measurement results are either equal or non-equal; in ordinal scale measurement in the case of non-equality the answer can become more specific, by means of the indication of the number of order positions separating the two values; finally, in interval scale measurement a distance between x_1 and x_2 is inherently, i.e., meaningfully, defined.

This peculiar kind of information is ordinarily taken into account in defining the criteria of compatibility between measurement results or of distortion acceptability when measurement results have to be compared to a reference value. Consider, e.g., a usual definition of compatibility between measurement results formalized as intervals: x_1 and x_2 are compatible if the distance between their central points is less than a given threshold value. It is manifest that this definition cannot be applied for ordinal or nominal scale measurement, for which metrical assertions are not in general meaningful. In the opposite case of nominal scale measurement, the very concept of interval is missing: x_1 and x_2 are simply subsets, and the definition of compatibility becomes the most general one, $x_1 \cap x_2 \neq \varnothing$, whose evaluation requires the check of all elements belonging to such subsets. The lack of information-from-structure could substantially increase the computational efforts needed in this case to control the compatibility. Analogously, the distinctions among statistics that can be applied to measurement results [23] substantially depend on the information-from-structure degree conveyed by such results. For example, in the case of nominal scale measurement the mode, but not the median or the mean, can be meaningfully applied, and in ordinal measurement the mean is still not meaningful.

R.Carnap [24] suggests the existence of an evolutionary path towards algebraically richer scales for any given measurand. In early stages man learns how to classify things with respect to the attribute, and therefore to measure in nominal scale; then, with the enhancement of the knowledge on the attribute itself, man learns how to order things with respect to it, and therefore to measure in ordinal scale; finally, sometimes man becomes able to define a meaningful metric among things with respect to the same attribute, and therefore to measure in interval, or ratio, or absolute scale. We believe that such an evolutionary path can be interpreted as the quest for an ever higher degree of information-from-structure.

5. Information-from-connection

What makes an evaluation pragmatically interesting is the possibility to substitute with the evaluated thing the corresponding symbols obtained in the evaluation, and therefore to formally treat symbols instead of empirically deal with things.

In the most elementary situations, such a possibility of elaboration is pre-theoretical, being based only on the properties of the scale in which the available value are measured. For example, having measured the geometrical dimensions of two things, a container and a potential contained thing, it becomes possible to establish with a purely symbolic procedure (thus avoiding any empirical operation) whether the latter thing can be actually contained in the former. The information-from-structure conveyed by such measurement results is such that a symbolic comparison of the kind "greater than" is indeed meaningful, in this case corresponding to what is empirically searched for, i.e., a truth value for the relation "contained in".

On the other hand, the symbolic treatment of the measurement results is often performed as an indirect measurement, i.e., on the basis of a "law" expressed as a relation among variables representing the values of measurable attributes. If the relation includes n variables and $n-1$ of them have been directly or indirectly measured, the relation itself can be applied to compute, i.e., indirectly measure, the value corresponding to the n-th variable. In this situation, the adequacy of the substitution thing-measurement result is thus affected by the validity of such a functional relation, and therefore is based on a theoretical hypothesis.

The existence of this kind of functional relations is such that measurement results, obtained in a direct or indirect way, convey some information with peculiar characteristics, due to the fact that measurable attributes are reciprocally connected through the functional relations: this component of the whole information brought by measurement results will be called *information-from-connection*.

The degree of the information-from-connection conveyed by a measurement result depends on "how much the measurand is connected with other attributes": the more the measurand is

connected (e.g., the greater is the number of functional relations in which the measurand is present), the higher is the information-from-connection degree conveyed by its values. In extreme cases, an attribute not entering in any functional relation, therefore not connected with any other attribute, is defined in a completely arbitrary way, and then pragmatically useless. As for the information-from-structure, the information-from-connection can thus be interpreted as a kind of nested information, for which any measurand value implicitly refers to all other attributes which the measurand is analytically related with.

It should be noted that the information-from-connection is not dependent on the information-from-structure. Although physical laws usually connect attributes that are measured in ratio (or interval) scale, some relations can be defined concerning only ordinal or even nominal attributes. In the nominal case, for example, relations have the form "if $x_1=x_2$ then $x'_1=x'_2$", where the x_i and x'_j are the values of two attributes. Such a relation expresses the hypothesis that if two things have the same value for the attribute x then they will have the same value also for the attribute x' (or, in set-theoretical terms, that the partition induced by the evaluation of x on the set of things is a refinement of the partition induced by the evaluation of x' on the same set). Although with different aims, an example in this perspective is presented and analyzed in its epistemological implications in [25]: the quality of an industrial product is largely an arbitrary attribute until evaluated only as perceived quality. On the other hand, its meaningfulness grows whenever it is shown how to derive it from a set of attributes related to production tolerances, i.e., when the degree of information-from-connection brought by measurement results increases.

6. Conclusions

Measurement and measurement systems can be analyzed according to different abstraction levels, from the physical layer that considers energy transfer through signals to various layers of information whose carriers are signals themselves:

* a *syntactic* layer, expressed here in terms of "information-from-selection", taking into account the purely set-theoretical fact that the measurement can be interpreted as a selection of elements

from a predefined set; the information is related here to the meaningfulness to discriminate elements that have been selected from those that have not been;

* a *semantic* layer, expressed here in terms of "information-from-structure", enriching the previous layer with the algebraic fact that different results of measurements for the same attribute can be compared / operated according to criteria dependent on the measurand scale type; measurement results are indeed values of morphisms, i.e., are related to measured things via suitable "meaning" functions and such functions conserve the relations defined among things;

* a *pragmatic* layer, expressed here in terms of "information-from-connection", in its turn enriching the previous layer by recognizing that the measurand is embedded in a network of attributes mutually linked by the relations in which they appear; measurement results are indeed values of measurands on which a body of knowledge is available.

The analyzed layers of information can be thus progressively adopted "for assessing the effectiveness of the measurement system as a means of acquiring knowledge of real world objects" [6]. The transition from one layer to the following one corresponds to an extension of the context in which the measurement results can be interpreted and of the nested information inherited by measurement results from such a context.

The limited emphasis that physical sciences traditionally put on semantics and pragmatics of their attribute qualification plausibly depends on a kind of "saturation effect": the most physical attributes are measured in interval, ratio, or absolute scales (i.e., they convey the "maximum" degree of information-from-structure), and are strongly related with each other via that body of knowledge that is physics itself (i.e., they convey the "maximum" degree of information-from-connection). In other terms, in characterizing the information brought by measurement results of physical attributes a single discriminating layer is often present, namely the syntactic one, related to the degree of uncertainty recognized in such results. A different situation can be met in the case of non-traditional attributes (the already mentioned "perceived quality" being in our opinion the best example of them: see [26]), for which the very problem of attribute definition is

often still a relevant issue, and also information-from-structure and information-from-connection are meaningfully modulated.

Finally, the analysis here presented is manifestly incomplete: the progressive enlargement of the context in which measurement results can be interpreted cannot stop at the various layers of information conveyed by *measurement results* themselves: it should encompass also the information *directly related the measurement* as an empirical operation. As a matter of fact, quality of measurement and quality of measurement results are related with each other, but they cannot be identified (for example, the response time of sensors influences the quality of the measurement system, and therefore of the measurement, but not in general of the measurement results). Measurement results are embedded in a (semantic and pragmatic) context, whose knowledge brings information on the results themselves: who the measurer is, when and with which system the measurement has been performed, how the influence quantities have been taken into account and estimated, …

The aggregation of such an information constitutes a measurement model large parts of which are available and employed before measurement to set up the measurement system. An effectively intelligent measurement system should be able to maintain and use such a knowledge base.

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Appendix. A tentative formalization

To formalize the different concepts of information that have been introduced here a purely set-theoretical standpoint will be adopted: compared to a first order logic approach based on model theory its notation is simpler, although in some respects more limited (for the current purposes, its most notable shortcoming is its inability to distinguish between the domain objects and the names by which they are denoted in a given language. In the following such a distinction is not however essential and will be left to the understanding of the reader).

The basic constructs of the formalization are relational structures and morphisms between them, as usual in the representational approach to measurement theories. A *relational structure* is an ordered pair $\mathbf{A}=\langle A,R_A\rangle$ of a domain set A and a set R_A of relations on A. The elements of R_A can have different arities: in particular 1-ary relations are individual constants of A, and n-ary operations on A appear in R_A as specific (n+1)-ary relations. A (homo)morphism from a relational structure $\mathbf{A}$ into a relational structure $\mathbf{B}$ is a rule $\mathbf{m}=\langle m,m_R\rangle$, $\mathbf{m}:\mathbf{A}\rightarrow\mathbf{B}$, that maps A into $m(A)\subseteq B$ and R_A into $m_R(R_A)\subseteq R_B$ such that $\forall r\in R_A$, $\forall a_i\in A$, $m_R(r)(m(a_1),\ldots,m(a_n))$ whenever $r(a_1,\ldots,a_n)$ (see [15] for a more extended presentation of these concepts).

A relational structure $\mathbf{E}$ is called *empirical* if its domain set E spans over the thing states under consideration; a relational structure $\mathbf{S}$ is called *symbolic* if its domain set S spans over a given set of symbols. An attribute is then formalized as a homomorphism $\mathbf{m}$ from an empirical relational structure $\mathbf{E}$ into a symbolic relational structure $\mathbf{S}$ (note that in the following the same symbol will be adopted for homomorphisms and their corresponding attributes).

The rationale of this formalization is that attributes are dealt with as representational tools: by means of them the information available on the measured things is extracted and expressed in symbolic way. The more the information on $\mathbf{E}$ is represented by the attribute in terms of $\mathbf{S}$, i.e., is conserved on $\mathbf{S}$ by the homomorphism $\mathbf{m}$, the more are the ways in which the symbols in S

can be processed in a meaningful way with respect to **E**, i.e., so that the results of such a processing can be back-propagated to E to infer a corresponding information for thing states.

Information-from-selection - In the most general case, only an equivalence relation r_{eq} is defined in R_E, such that $m_R(r_{eq})$="=" (the logical symbol of identity, implicitly assumed as always present in both R_E and R_S and for which the usual infix notation is here adopted), i.e., $r_{eq}(e_1,e_2)$ iff $m(e_1)=m(e_2)$. If the complete specificity of the attribute is assumed, the evaluation of m on a thing state e corresponds to the selection of the symbol $s=m(e)$ from the set S. If back-propagated to **E** this information only allows to know that the evaluated state belongs to the r_{eq}-equivalence class $m^{-1}(s)$ and not to any other element of the r_{eq}-partition set (in the case of non complete specificity $m^{-1}(s)$ is a compatibility class: with respect to the information conveyed by the attribute the generalization is trivial). Such an information has been called here *information-from-selection*, and can be quantified by the (Hartley-)Shannon measure.

Information-from-structure - Let us now assume a more specific form for the empirical relational structure, namely $\mathbf{E}=<E,R_E>$ such that: $E=E^*\cup E'$ where E^* is called a *standard set* (a more general concept of the usual "standard *sequence*") and E' is a set of thing states as previously stated; $R_E=R_{E^*}\cup R_{E'}\cup\{r_{eq}\}\cup\{r'_{eq}\}$ where R_{E^*} and $R_{E'}$ are (possibly empty) sets of relations defined on E^* and E' respectively, and r_{eq} are r'_{eq} are equivalence relation defined on $E^*\times E'$ and E' respectively. Let m^* and m^*_R be the restrictions of m and m_R on E^* and R_{E^*} respectively.

While E' is empirically given and not under the control of the measurer, who is indeed interested in acquiring some knowledge on it, the standard set E^* is chosen so that $\mathbf{m}^*=<m^*,m^*_R>$ is an homomorphism from $\mathbf{E}^*=<E^*,R_{E^*}>$ into $\mathbf{S}$ and m^* is injective (the resolution of the attribute is indeed defined by E^*). Moreover, the standard set E^* is chosen so that the equivalence relation r_{eq} can be evaluated as an empirical comparison between any given thing state e' and a given standard set element e^*, and $\forall e'\in E'$, $\exists!e^*\in E^*$ such that $r_{eq}(e^*,e')$, a condition expressing the completeness of the standard set E^* for E'.

The equivalence relation r'_{eq} between thing states is then defined so that $\forall e'_1, e'_2 \in E'$, $r'_{eq}(e'_1, e'_2)$ iff $\exists e^* \in E^*$ such that $r_{eq}(e^*, e'_1)$ and $r_{eq}(e^*, e'_2)$ (the uniqueness of e^* follows from the injectivity of m^*).

The basic result here follows: $\forall e' \in E'$, $m(e') = m^*(e^*)$ where $r_{eq}(e^*, e')$.

Furthermore, via r_{eq} the relations in R_{E^*} are inherited by the thing states in E': for any given $r^* \in R_{E^*}$, there exists one and only one corresponding induced relation $r' \in R_{E'}$ such that if $r^*(e^*_1, e^*_2)$ (e.g., e^*_1 follows e^*_2 in an empirical order), and $r_{eq}(e^*_1, e'_i)$ and $r_{eq}(e^*_2, e'_j)$ then $r'(e'_i, e'_j)$ (e'_i follows e'_j in the corresponding empirical order).

Provided that a proper identifier ref^* is assigned to the standard set E^* (e.g., "Richter scale", or "Celsius degrees", or "millimeters"), a measurement result of the thing state e for the attribute $\mathbf{m}$ is expressed as customarily as $s\ ref^*$ where $s = m(e) = m^*(e^*)$ and $r_{eq}(e^*, e)$.

The different scales of measurement are obtained by suitably characterizing the set R_{E^*}. For example, an attribute is measured in nominal scale if $R_{E^*} = \varnothing$, and in ordinal scale if the only element of R_{E^*} is a total order. In any given scale the range $\mathbf{S}$ is identified but the automorphism group of its admissible transformations. Hence in the expression of a generic measurement result the two terms s and ref^* convey two qualitatively different kinds of information: while, as previously considered, the former is the result of a selection, the latter represents the structure of the relation set in which the former has to be interpreted. In addition to the information-from-selection brought by s, if back-propagated to $\mathbf{E}$ this information allows to know how to operate with different thing states with respect to the measurand, i.e., which empirical relations and operations can be meaningfully applied to them. Such an information has been called here *information-from-structure*, and allows to compare any two attributes according to the partial order defined in terms of set inclusion between the automorphism groups associated with their scales. An attribute $\mathbf{m}$ conveys more information-from-structure (of course an ellipsis for "the measurement results for an attribute $\mathbf{m}$ convey more information-from-structure") than an attribute $\mathbf{m}'$ whenever the automorphism group of $\mathbf{m}$ is a subgroup of the automorphism group

of **m**'. From this general definition the known "evolutionary" relation among nominal, ordinal, interval, and ratio scales trivially follows, being the automorphism groups of 1-1 transformations (i.e., permutations), monotonic increasing transformations, positive linear transformations, and similarity transformations included each in the previous one.

Information-from-connection - Let us now assume that a set $\{\mathbf{m}_i\}$, $i=1,\ldots,k$, of $k\geq2$ attributes is defined such that $\mathbf{m}_i:<E,R_{E,i}>\rightarrow<S_i,R_{Si}>$. Each of such attributes maps therefore the same set E of thing states into a set S_i of symbols (these sets S_i can be in principle distinct, although this is not a requirement: as previously considered, what actually characterizes the attribute $\mathbf{m}_i$ is the set R_{Si} of the relations defined on S_i, and not S_i itself).

Among the k attributes $\mathbf{m}_i$ let a functional relation f be defined, $f:m_1(E)\times\ldots\times m_{k-1}(E)\rightarrow m_k(E)$, such that for any given thing state $e\in E$ the attribute value $m_k(e)$ can be computed as $f(m_1(e),\ldots,m_{k-1}(e))$. This functional dependence corresponds to an indirect measurement, in which the knowledge of the values of some attributes conveys an information on the measurand: if back-propagated to **E** this information allows to infer the value of an attribute from the values of a set of attributes "connected" to it via a functional relation. Such an information has been called here *information-from-connection*, and allows to compare any two collections of mutually connected attributes in terms of the number of independent functional relations defined among the attributes. The information-from-connection brought by an attribute $\mathbf{m}_i$, $i=1,\ldots,k-1$ on the attribute $\mathbf{m}_k$ (of course an ellipsis for "information-from-connection brought by the measurement results for an attribute $\mathbf{m}_i$ on the measurement results for the attribute $\mathbf{m}_k$) increases as the number of independent functional relations in which both $\mathbf{m}_i$ and $\mathbf{m}_k$ appear increases.

Finally, let us remark the very general fact that the information-from-connection that an attribute conveys on its connected attributes cannot increase the information-from-structure brought by them. Indeed, in the extreme case the function f acts as a simple classifier (i.e., "forgets" all the

relations in the sets R_{Si}) the information-from-structure possibly conveyed by $\mathbf{m}_k$ derives solely from other methods of measurements applied to it.