

A structural model of direct measurement

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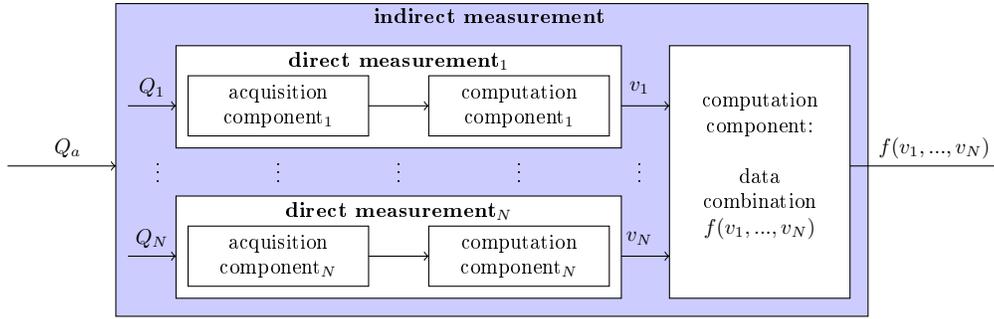
Abstract

In the last few decades foundations of measurement have developed so as to account for both the role of modeling *in* measurement, in particular relating to the presence and the effects of measurement uncertainty, and the fact that any measurement is performed by using instruments that work on the basis of transduction effects and provide justified results only in so far as they are properly calibrated. This has triggered a new interest about the role of instruments in the models *of* measurement. The structure of the process has been variously studied in reference to the connection between measured properties and indications provided by instruments, and to the way in which intersubjective information on the measurand is acquired through instrument calibration. From such a background this paper proposes a comprehensive structural model of direct measurement, whose functional elements, empirical and informational, are presented with a bottom-up strategy as a set of interrelated modules. The result, shown to be a generalization of some of the models currently available in the literature of measurement science, highlights the key role of scales for measurement, clarifies the conceptual and operative relations between direct measurement and calibration, and identifies the principal sources of measurement uncertainty in the structural context of the process. This model is intended to interpret both physical and non-physical measurements, and as such it is a component of a “measurement across the sciences” research programme.

Keywords: Structural model; Foundations of measurement; Transduction; Calibration; Uncertainty.

1 Introduction

Measurement aims to attribute a value to an empirical property intended to be measured [18, def. 2.1]. As such, it is a process that includes both *acquisition* components, which through empirical processes produce information on a given set of properties, and *computation* components, which exploit such information in order to produce the measurement result. This task can be achieved by means of direct or indirect methods of [measurement](#). In *direct methods*, measuring instruments are designed so as to directly interact with the object under measurement relative to the measurand. In *indirect methods*, by contrast, the information on the measurand is inferred from [information about](#) other properties, which ultimately are measured in a direct way. Computation components are used to return values both on properties which are directly measured, typically on the basis of transduction laws modeling the transduction effect on the basis of which the measuring instrument operates, and on properties which are indirectly measured, by combining data on directly measured properties according to specific mathematical laws. This general structure can be then represented as in the following diagram, where Q_a is the measurand, Q_1, \dots, Q_N are intermediate measurands, i.e., the properties on which Q_a depends, v_1, \dots, v_N are the values of these properties, and $v = f(v_1, \dots, v_N)$ is the value attributed to Q_a .



Hence measurement either is direct or includes one or more direct measurements as its core components. In accordance with this insight, the present paper aims *to develop a structural model of direct measurement*. To this purpose we build on one of the most significant achievements of the last decades in foundations of measurement: the acknowledgment that the justification of the quality of measurement results [13, 22, 26, 33, 36] is to be understood in reference to *the structure of the process*: “black box”, i.e. input-output, models are not sufficient to account for the trustworthiness of measurement [24], that must be then modeled by “opening the box” and studying the process in terms of its components and their relationships. [The importance of modeling measurement is witnessed by several works that in the recent years have enriched the literature of measurement science with proposals on this subject, including the foundational papers by Finkelstein \[1, 8, 9\] and Sydenham \[35\], and more recently, the analytical contributions by Morawski \[27\], Rossi \[30, 31\], Ruhm \[32\], and also the present authors \[11, 13, 21\].](#)

The model proposed here builds on these works and is expected to generalize them and provide the conditions for:

- justifying in structural terms the degree of objectivity and the intersubjectivity of measurement results, by identifying different sources of uncertainty and helping in estimating their relative import;
- clarifying the analogies and the differences between measurement and calibration of measuring instruments, and explaining the claim that calibration is necessary for measurement;
- allowing for a comparative analysis of measurement of physical and non-physical properties and highlighting the invariant aspects of measurement across physical and non-physical applications.

Moreover, as measurement is based on a number of modeling activities, a structural model of direct measurement works as a meta-model providing the conditions for identifying where modeling activities are explicitly or implicitly performed and what kinds of models are at stake.

Remark 1. At least since the publication of the *Guide to the expression of uncertainty in measurement* (GUM) [16] the term “measurement model” has been used also to refer to the mathematical relation by which a value of the measurand is computed from the values of some “input quantities” and the uncertainties of such input quantities are propagated to obtain an uncertainty for the measurand [18, def. 2.48]. In this sense, a measurement model is a component of the measurement process (typically the computation component in the diagram above), not an interpretation of what a measurement is and how it is structured, as instead intended here, in line with the usual meaning of *model of X as interpretation of X*.

We introduce the model with a step-by-step, in fact module-by-module, strategy, where each module models a sub-process that is conceptually and functionally independent of the other ones, so that the justification of the whole model only depends on the justification of its components and the way they are combined. In doing so, we start from two sets of assumptions: the first set concerns the measurand, i.e., the property intended to be measured, and constitutes a basic theory of properties which is necessary to provide a general framework for understanding measurement; the second set concerns the measurement, as an activity performed by human beings who aim to

acquire information on empirical objects, intended in a wide sense, including physical bodies, empirical phenomena, and empirical processes.

1.1 Assumptions on the measurand

The attribution of values to measurands (as in the definition of ‘measurement’ given by the *International Vocabulary of Metrology* (VIM), [18, def. 2.1]) is informative inasmuch as values represent properties and their empirical relations [7, 19, 28, 34]. Hence, nowadays it is widely accepted that a theory of measurement is based on a *theory of properties* that accounts for both the distinction between general and individual properties, e.g., temperature and the temperature of a given object respectively, and the representation of properties of objects in terms of values. In particular, as to the distinction between general and individual properties we assume that [25]

- individual properties, and thus properties of objects, are instances of general properties (the temperature of a given fluid is an instance of the general property temperature);
- properties may be defined, and the definitions may involve empirical and theoretical components (a measurand might be defined as the temperature of a given fluid in given conditions of the fluid and the environment);
- general properties may be classified according to the relations that are known to be invariant among their instances (in the past temperatures were compared only by order, while thermometry made them comparable also in an interval scale [34]).

In what follows, we will use capital letters, like P , to refer to general properties and capital letters with indices, like P_i , to refer to their instances. It is worth noting that instances of general properties can be referred to in different ways. Thus, an instance of [speed](#) can be referred to *by name*, i.e. by using a specific expression (e.g., c), or *by description*, i.e. by specifying entities that support the property (e.g., the [speed](#) of light in vacuum), or *also by value*, i.e. by specifying an element in a classification (e.g., 299 792 458 m/s).

As to the representation of properties of objects by values of properties, we assume that [19]:

- both properties of objects and values of properties are individual properties: properties of objects are identified by reference to an object, whereas values are identified by reference to a set of which they belong (both the temperature of a given fluid and 300 K are temperatures, the former identified as a property of that fluid, the latter as the temperature identified as 300 times the unit temperature in the scale K);
- distinct values are attributed to distinguishable properties of objects of the same kind in such a way that the mathematical relations among values both imply and are implied by the invariant relations among the corresponding properties of objects (a greater value of temperature is attributed to a warmer fluid);
- the choice of the system of values adopted to represent properties of objects is unique only up to a group of transformations satisfying the condition that the representation preserves the invariant relations (different scales of temperature can be defined, with the appropriate transformation rules to convert values on one scale into values on another one).

In summary, values are informational entities that operate as identifiers for properties of objects, so that a relation such as $\Theta_a = \theta_i$, where Θ_a is the temperature of the object a and θ_i is a value of temperature like 293.15 K, asserts that the object a has a temperature which is identified as the temperature that is 293.15 times the temperature identified as K.

On this basis let us consider the assumptions concerning measurement.

1.2 Assumptions on measurement

Measurement is considered here *a model-based, partly empirical and partly computational, process aimed at acquiring sufficiently object-related (i.e., objective) and subject-independent (i.e., intersubjective) information on empirical properties of objects by using suitable instruments, and at presenting such information in terms of justified attribution of values to properties by using suitable statement systems.*

The key components of this characterization are that measurement:

- attributes values to empirical properties of objects, and as such maps empirical entities to informational entities;
- is a designed-on-purpose, not a natural, process, based on models built or assumed according to the given purpose;
- aims to acquire information that is sufficiently object-related and subject-independent with respect to the given purpose;
- reports the acquired information in a way that is justified with respect to the given purpose.

A model of measurement is expected to explain why and how measurement results are justified. In the case of direct measurement this requires the analysis of the structure of the process, including the comparison between the property under measurement and the properties identified as elements of a scale, i.e., between a property known by description and some properties known by value [25], and thus the calibration of the instrument against a reference to which the results will be metrologically traceable [18, def. 2.41].

1.3 Schema of the study

Given the complexity of the subject and our ambition to define a model that is applicable to all cases of direct measurements, in what follows we propose a bottom-up presentation, that starts from the simple model of a specific intermediate process, called “pre-measurement” [11], *i.e., the process performed by a properly operated but still not calibrated measuring instrument. By first analyzing pre-measurement we are able to discuss the features of measurement that are independent of metrological traceability, thus highlighting the conditions that allow producing results that may be objective but cannot be intersubjective yet.* On this basis we build to develop first a model of the ideal structure of measurement, which already includes all the components of the process but still assumes that the process is stable and the role of influence properties can be neglected. Hence such an ideal model does not take any source of uncertainty into account. These sources are then considered in a further model, whose generality we prove by also showing that it embeds several other models, including representational ones, previously proposed in the scientific literature. This generality is also highlighted by the fact that the presentation does not rely on any specific algebraic structure of the property under measurement, that only as a specific case is a ratio quantity (*hence we use the term “scale” for continuity with the tradition, at the same time allowing in principle even non-ordered scales, semantically a questionable option;* whether non-quantitative properties can be measurable is an issue that we do not discuss here; for a metrological analysis of the subject see [23]).

Direct measurement is modeled here so that its key component is a measuring instrument, with the fundamental task of implementing what has been called the *unobservable-observable bridge* [4, p.86]. With the aim of maintaining the focus on the role of the instrument, we assume here that the property with which the instrument interacts is the property intended to be measured, i.e., the measurand, and we only mention, in section 8, the problems arising from the definition of the measurand and the coupling of the object under measurement and the instrument.

In order to stress the structural nature of the model, and after a preliminary characterization of the whole process (section 2), we introduce the model as a system of interacting modules, where each module is formally a function,

written here as a triple $\langle \text{domain}, \text{codomain}, \text{mapping} \rangle$ for clarity: a *classification module* made of a *matching module* and a *private scale module*, to be configured before performing the measurement (section 3), a *transduction module* whose operation leads to pre-measurement (first model, ideal pre-measurement: section 4), a *calibration module* based on a *public scale module* (section 5), and a *measurement module* (second model, ideal measurement: section 6). This allows us to compare this model with other models presented in the scientific literature (section 7). By exploiting this structure, the condition of ideality is finally removed, leading to the proposed model (third model, actual measurement: section 8).

For making the presentation as concrete and simple as possible, we exemplify each module through a case of physical measurement. At the end of the paper, other examples, of both physical and non-physical measurements, are mentioned for highlighting the generality of the proposed model.

2 Preliminary characterization

The measurement of temperature by means of a usual mercury thermometer is the example on which we develop the analysis that follows: it is both sufficiently simple to allow us to avoid insignificant details, and sufficiently complex to instantiate all the fundamental characteristics of a direct measurement (this choice is also made in [31, 33]). Such an instrument is designed to produce distinct positions of the upper surface of the mercury column as an effect of its thermal interaction with objects of sufficiently distinct temperatures. These positions can be associated with marks made on the surface of the glass capillary, related to distances with respect to a conventional zero mark and corresponding to distinct temperatures. Thus, the matching of the upper surface of the mercury with a mark corresponds to a distance between the mark and the zero mark, which in turn corresponds to a temperature, as obtained via instrument calibration. The steps for calibrating the thermometer are then:

1. select a set of known temperatures and a set of objects each having a temperature in such a reference set, [then such that the value of the temperature of each object is assumed to be known](#);
2. make the device interact with each object of the set;
3. for each interaction, record the [instrument](#) indication, i.e., the [mark corresponding to the](#) position of the mercury, together with the value of the corresponding reference temperature

(this is a generic procedure: if the mapping [thermometer marks](#) - temperatures is assumed to be a parametric analytical function, then a number of interactions equal to the number of parameters is sufficient). On this basis, the steps for measuring the temperature of an object by means of a calibrated thermometer are:

1. make the device interact with the object whose temperature is to be measured and thus obtain the mercury flow to a given position in the capillary;
2. match the position of the mercury with the indications recorded in calibration;
3. attribute as measurement result the value of the reference temperature of the matched indication.

The whole system can be then interpreted as a device implementing a function which takes *the temperature of an object* in a certain domain and returns *a value of temperature*: this empirically determined correspondence between the temperature of an object and a value of temperature constitutes the measurement result, where the acknowledgment of a non-null measurement uncertainty only requires to extend this interpretation.

Remark 2. [The distinction between direct and indirect methods of measurement is not uniquely understood, and it might be argued that the operation of a mercury thermometer is an example of an indirect method.](#) The first edition of the VIM, for example, mentions the measurement of a temperature using a resistance thermometer as a

case of indirect method of measurement, defined as “a method of measurement in which the value of a measurand is obtained by measurement of other quantities functionally related to the measurand” [14, def. 2.14]. The weak point of this characterization is that only few cases of direct measurement would be actually possible (interestingly, the subsequent two editions of the VIM do not have entries for ‘direct method’ and ‘indirect method’, perhaps as an implicit acknowledgment of this weakness). The examples that the VIM1 proposes are only “measurement of a length using a graduated rule” and “measurement of a mass using an equal-arm balance”, and the latter is already questionable, given that it could be described as the measurement of a mass via its functional relation with the moment of a gravitational force, and therefore as indirect. The GUM comments that “in most cases, a measurand Y is not measured directly, but is determined from N other quantities X_1, X_2, \dots, X_N through a functional relationship $f: Y = f(X_1, X_2, \dots, X_N)$ ” [16, 4.1.1]. As introduced above, the distinction between direct and indirect methods of measurement worth to consider is instead between (i) cases in which measuring instruments are used that are sensitive to individual properties of the same kind as the measurand and (ii) cases in which the information on the measurand is instead inferred from data empirically acquired about properties other than the measurand, where then this inferential component adds complexity to the process. In this sense, *direct methods are the core of any measurement process*.

3 Instrument configuration

The purpose of any measurement is to establish a correspondence between *properties of objects* in given domains and *values of properties* in given sets of values, thus generating a link between *the empirical realm* of properties of objects and *the informational realm* of values of properties (note that this is not as obvious as it could seem, and it is in fact in opposition to the purely mathematical modelings according to which measurement maps values to values – see, e.g., [3, p.3]). The core component for producing this correspondence is what we call a *classification module*, which indeed takes properties of objects as input and returns values as output. This module is in turn based on the composition of a *matching module*, that compares properties to be classified with reference properties, and a *private scale module*, that associates reference properties to values. **Let us introduce these modules in the case of a mercury thermometer.**

3.1 Instrument configuration: private scale module

Some positions in the capillary of the thermometer are marked and associated to a set of values used to discern and identify them. The marked positions are the empirical outcomes that the instrument is designed to make discernible. These outcomes are the *instrument indications* and the values associated to the marks the *instrument indication values* (note that this is not compliant with [18, def. 4.1], which considers indications as values, but does not give a name to the property evaluated by such values). This **key process** is formalized by

- a set $\{P_j^*\}$ of empirically distinguishable properties – the instrument indications – that are instrument-specific, in this case marks **on the thermometer capillary** as reference positions relative to a conventional zero position;
- a set $\{p_j\}$ of identifiers – the instrument indication values – that are again instrument-specific, in this case values attributed to marked positions;
- an *identification function* $\iota_P : \{p_j\} \rightarrow \{P_j^*\}$, such that $P_j^* = \iota_P(p_j)$ is the indication identified by the indication value p_j ; since the set $\{p_j\}$ is chosen so as to make each indication uniquely identifiable, this function is bijective and instrument-specific (e.g., **the first indication might be identified by the value p_1 , the second indication by the value p_2 , and so on**).

The inverse of the identification function ι_P models the recognition of the indication value that identifies each indication P_j^* . Hence it is a *recognition function* $\rho_P : \{P_j^*\} \rightarrow \{p_j\}$, such that $p_j = \rho_P(P_j^*)$ is the indication value recognized for the indication P_j^* .

$$\{P_j^*\} \begin{array}{c} \xrightarrow{\rho_P} \\ \xleftarrow{\iota_P} \end{array} \{p_j\}$$

The $\iota_P - \rho_P$ functional loop is crucial for measurement, since it constitutes the *link between the empirical realm and the informational realm* in the model of the instrument behavior: the p_j 's provide information on the P_j^* 's. This is the fundamental justification for the claim that a measuring instrument has inherently both an empirical and an informational component. In particular, the set $\{p_j\}$ of values is chosen so that the identification function ι_P relates values to entities which are empirically determined, i.e., the reference positions selected by putting marks on the capillary. A triple like $\langle \{p_j\}, \{P_j^*\}, \iota_P \rangle$ is a *scale*, in which a function relates informational and empirical entities with the purpose of identification (this is in analogy with the definition in the representational theories of measurement – see [28, p.54], where the function in the scale is instead identified with ρ_P , i.e., the inverse of ι_P), and in this case a *private scale module*, since *instrument-specific*.

Definition 1. (Private scale module): private-scale $\stackrel{def}{=} \langle \{p_j\}, \{P_j^*\}, \iota_P \rangle$

The private scale is constructed by mapping values, i.e., intersubjectively discernible and transferable *informational entities*, to reference properties of the measuring instrument, i.e., intersubjectively discernible *empirical entities*, via the identification function ι_P . The injectivity condition on ι_P is a consequence of the fact that the reference properties in the scale are chosen in such a way that they are discernible, so that distinct values are mapped to distinct reference properties in order to preserve distinction, and therefore information. The surjectivity condition on ι_P is a consequence of the still more basic fact that the reference properties in the scale are chosen, and it would be idle to choose discernible reference properties and then not to exploit them. *Each mark on the capillary is then uniquely identified by one value, e.g., first mark $\leftrightarrow p_1$, second mark $\leftrightarrow p_2$, and so on.*

Once a private scale has been constructed, it is used by mapping the reference properties of the measuring instrument to values via the recognition function ρ_P . Under the hypothesis that the instrument is stable, the reference positions P_j^* are themselves stable and therefore *the private scale provides a reliable instrument-related link between the empirical and the informational realm*. As we will see, every function in the model linking these two realms results from the composition with a scale.

Remark 3. The properties under measurement can be discovered to be comparable by more than just discernibility. Whenever the comparison is invariant, for example, by order or by ratio, measuring instruments can be designed that produce indications from an analogously structured set. In this case, the set of indication values is chosen so as to preserve such a structural information, a condition mathematically described in terms of morphisms in representational theories of measurement [19]. Therefore, it is not surprising that, in identifying a structured set of reference properties, numerical systems are used to specify the sets of values: this preserves not only distinctions between properties, but also their ordinal and quantitative relations.

3.2 Instrument configuration: matching module

The private scale is *constructed in such a way* that any position P_m of the upper surface of the mercury – the index m highlights that such a position is a property of the measuring instrument – can be associated to an element of the private scale, i.e., the indication P_j^* , that best matches P_m . The instrument is then partly characterized by

- a set $\{P_m\}$ of observable properties, where each *observable property can be in principle matched with one marked indication* of the private scale;

- a *matching function* $c_P : \{P_m\} \rightarrow \{P_j^*\}$, such that $P_j^* = c_P(P_m)$ is the indication that best fits with the property P_m ; c_P is typically a many-to-one function.

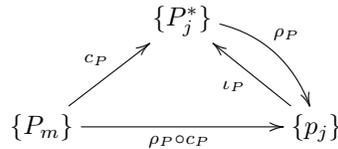
The relation $P_j^* = c_P(P_m)$ states the empirical fact that the reference position P_j^* is the indication corresponding to the position P_m . We call the triple $\langle \{P_m\}, \{P_j^*\}, c_P \rangle$ a *matching module*, and each P_m is said to be observable because it can be matched with a reference position which is discernible, and so recognizable through the private scale. Hence, [observability is intended here not as theory-independence, but in a relative sense, i.e., as being recognizable and exploitable](#) in a procedure allowing us to measure something else (about the role played by measuring instrument to make properties observable, see [4]).

Definition 2. (Matching module): $\langle \{P_m\}, \{P_j^*\}, c_P \rangle$

The fact that c_P is usually a many-to-one function can be in particular acknowledged if general properties with an uncountable set of instances are admitted. In this case there is no way to construct a one-to-one match between $\{P_m\}$ and $\{P_j^*\}$, since $\{P_j^*\}$ is a human construction and it is not possible for human beings to construct a uncountable set of empirical elements. The set $\{P_j^*\}$ is chosen precisely in virtue of the assumption that any observable position P_m of the upper surface of the mercury can be observed and matched with the [empirical](#) elements of the scale, *i.e., the marks on the capillary*. This is an *observability condition* on the instrument, which usually implies the quantization of the property relative to the instrument scale. [The matching](#) is a process that can be performed automatically by a device operating as a quantizer or, as in our example, by the measurer.

3.3 Instrument configuration: classification module

By composing the matching function c_P and the recognition function ρ_P , a function which attributes values in the instrument-specific, private scale to observable properties is obtained.



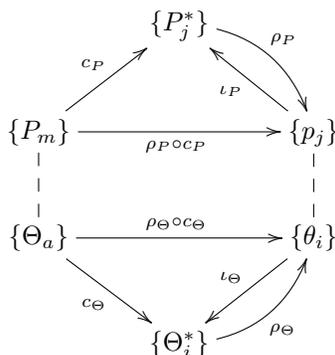
The relation $p_j = \rho_P \circ c_P(P_m)$ states the empirical *and* informational fact that the value corresponding to the indication P_j^* is the value corresponding to the observed position P_m . We call the triple $\langle \{P_m\}, \{p_j\}, \rho_P \circ c_P \rangle$ a *classification module for P*. It is worth noting that the classification module might include auxiliary classification modules and transduction modules, but these are implementation details that do not modify the basic structure we are proposing.

Definition 3. (Classification module for P): $\langle \{P_m\}, \{p_j\}, \rho_P \circ c_P \rangle$

The classification module wraps the matching module, which determines the relevant component of the state of the instrument, and the private scale module, which encodes the state component with an identifier, and therefore is the informational core of the measuring instrument. It conveys, as we will see later, explicit and implicit information

1. on the instrument scale;
2. on the measurement principle [18, def. 2.4], characterized by the effect of the measurand on the property P ;
3. on the measurement procedure [18, def. 2.6], characterized by the measurement principle and the matching procedure.

By coupling this instrument-related classification module, having at its core a private scale of lengths, with a structurally similar classification module, whose core is a public scale of temperatures, the measurement of temperatures by means of the thermometer becomes possible, as an instrument-mediated process that associates temperatures Θ_a to values of temperature θ_i . By anticipating the outcomes of the analysis that we are going to develop, the whole system can be depicted as in the following figure, where, in symmetry with the upper triangle, the lower triangle includes measurands Θ_a , reference temperatures Θ_i^* , and values of temperature θ_i .



Despite its complexity, the symmetric structure provides some insights into two key features of measurement:

- it connects a private scale (upper side of the diagram) and a public scale (lower side); the connection is created by the empirical cause-effect relationship between temperatures to be measured Θ_a and instrument **observable properties** P_m : this provides a condition for the justification of the hypothesis that the connection is reliable, and therefore that measurement results are trustworthy;
- it connects an empirical component (left side) and an informational component (right side): the connection is created by the private scale and the public scale, which are built under controlled conditions: this provides another condition for the justification of the hypothesis that the connection is reliable, and therefore again that measurement results are trustworthy.

The next sections introduce the second, temperature-related, classification module and on this basis a model of ideal direct measurement is obtained.

4 First model: ideal pre-measurement

A mercury thermometer is sensitive to temperature in such a way that, when it is properly put in interaction with an object, it changes its state in function of the temperature of the object, with the mercury flowing in the capillary and changing the position of its upper surface. After the transient, the steady state position allows us to represent, via the private scale of the instrument, the temperature of the object by means of a value. Since the obtained information is instrument-specific, and then lacks the condition of intersubjectivity which characterizes measurement [22], we call this process *pre-measurement* [11]. In addition, since measurement uncertainty is not taken into account yet, what is presented in this section is an *ideal* model of pre-measurement.

4.1 Ideal pre-measurement: transduction module

In the interaction with an object a , with temperature Θ_a in a set $\{\Theta_a\}$ of measurable temperatures, the mercury in the thermometer changes its volume, so that the position of the mercury upper surface becomes P_m . This is a physical process, based on a transduction effect, in this case of thermal expansion of bodies. The instrument is then partly characterized by the triple $\langle \{\Theta_a\}, \{P_m\}, f_P^\Theta \rangle$, where

- $\{\Theta_a\}$ is a set of properties that are inputs of the transduction function, in this case a set of temperatures that given objects can have;
- $f_P^\Theta : \{\Theta_a\} \rightarrow \{P_m\}$ is a *transduction function*, such that $P_m = f_P^\Theta(\Theta_a)$ is the property that results from transducing the property to be measured by means of the instrument; f_P^Θ is typically a many-to-one function.

The relation $P_m = f_P^\Theta(\Theta_a)$ states the *empirical fact* that the temperature Θ_a has been transduced into the position P_m , so that P_m is assumed to be an effect of Θ_a , mediated by the thermometer and where the context is left implicit. In this sense, P_m is *the transduced property*, as obtained by a transduction effect *chosen so as* to make the transduced property observable. We call the triple $\langle \{\Theta_a\}, \{P_m\}, f_P^\Theta \rangle$ a *transduction module*.

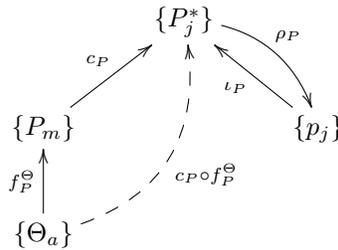
Definition 4. (Transduction module): $\langle \{\Theta_a\}, \{P_m\}, f_P^\Theta \rangle$

The simplest model assumes that P_m depends only on Θ_a , thus considering the instrument to be perfectly selective and stable, and as such independent in its behavior of any influence property, such as environment pressure: *whenever bodies of the same temperature interact with the thermometer, the same position of the mercury in the capillary is obtained as the result*. This makes the transduction module an appropriate model of the assumed deterministic transduction.

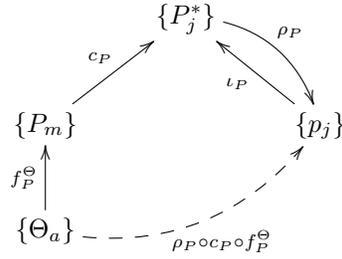
Remark 4. The function f_P^Θ is the cause-effect relationship assumed to exist between properties of objects and *observable properties* of the instrument, which are empirical entities. It is not assumed that the analytical form of *this function* is known. However, even in the case its analytical form were assumed to be fully known, for example as a linear dependence of P_m on Θ_a with known parameters (thus reflecting an ideal behavior of the transducer operating under ideal conditions), its output would remain an *empirical* entity. For producing *information* on Θ_a in the form of values, the instrument scale must be exploited.

4.2 Ideal pre-measurement: evaluation

The instrument is designed so as to transduce an input property into an output property that is observable, in the sense discussed above, and therefore so that the input temperatures Θ_a can be associated to the indications P_j^* via the composed function $c_P \circ f_P^\Theta$.



The relation $P_j^* = c_P \circ f_P^\Theta(\Theta_a)$ states the empirical fact that the reference position P_j^* is the matched indication (*hence a given mark on the capillary*) corresponding to Θ_a . In fact, the aim of the instrument is to transduce an input property into an output property in such a way that distinctions in the properties under measurement can be inferred from observable distinctions in the indications. In addition, by composing $c_P \circ f_P^\Theta$ with the recognition function ρ_P that results from the instrument configuration a pre-measurement is obtained.



The relation $p_j = \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$ states the *empirical and informational fact* that p_j is the value that identifies Θ_a in the instrument-related private scale. In pre-measurement, this allows us to obtain a private value for the property under measurement, a conclusion that is expressed by the following

Claim 1. Θ_a is represented by p_j in **private-scale** if and only if $p_j = \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$.

Thus, until instrument-related information is acceptable, temperatures are evaluated with respect to the private scale, and are then associated to values of position.

4.3 The stages of the ideal pre-measurement

The previous analysis shows that an ideal pre-measurement is a process performed through the following stages:

1. the preliminary construction of a private scale, which is *conventional*, even though constrained by the available knowledge about the features of the measuring instrument;
2. the application of a transduction module, which is an *empirical* process based on physical causation, and so it is to be accurately modeled in order to estimate its import on the information that will be produced;
3. the application of a matching module, which is basically an *empirical* process of quantization, and so it is to be accurately modeled in order to estimate its import on the information that will be produced;
4. the application of the private scale, which is a *theory-based* process, since the recognition function ρ_P , initially just the inverse of the identification ι_P , is computed at subsequent stages under the hypothesis of the stability of the instrument.

According to this model, a pre-measurement is able to produce *object-related*, i.e., objective, *information*. Whenever private information is sufficient, the fact that a pre-measurement conveys information on the temperature Θ_a in terms of a value of position p_j in **private-scale** is not a problem.

5 Instrument calibration

While pre-measurement results are non-transferable, because instrument-specific and therefore private, measurement aims at producing information which is not only object-related – something that ideal pre-measurement guarantees – but also *subject-independent*, i.e., intersubjective, because instrument-independent and therefore publicly interpretable. To this goal measurements performed according to a direct method require the measuring instruments to be calibrated against a public scale.

5.1 Instrument calibration: public scale

Some objects are chosen, in a set of easily accessible or reproducible objects, whose temperatures are sufficiently stable, so that such temperatures can be identified by reference to them. We call these objects *reference objects* or also, more customarily, *measurement standards*, their temperatures *reference properties*, and the values associated to them *reference values*.

Remark 5. In the case the property under measurement is embedded in an algebraically sufficiently rich structure, measurement standards are produced as realizations of the definition of the unit of that property, and of some multiples or submultiples of the unit, either as primary realizations or derived by existing realizations through calibration in a traceability chain. [This is the case of the historical development of the measurement of temperature \[5\], in which sometimes only two reference temperatures have been chosen \(e.g., of the melting and the boiling points of water\).](#)

As in the case of the private scale, this can be formalized by a triple $\langle \{\theta_i\}, \{\Theta_i^*\}, \iota_\Theta \rangle$, where

- $\{\Theta_i^*\}$ is a set of empirically distinguishable properties, the reference temperatures, that are public references of temperature;
- $\{\theta_i\}$ is a set of identifiers, that are reference values attributed to temperatures;
- $\iota_\Theta : \{\theta_i\} \rightarrow \{\Theta_i^*\}$ is an *identification function*, such that $\Theta_i^* = \iota_\Theta(\theta_i)$ is the reference property identified by the value θ_i ; since the set $\{\theta_i\}$ is chosen so as to make each indication uniquely identifiable, this function is bijective and instrument-independent.

The inverse of the identification function ι_Θ models then the recognition of the value that identifies each reference property Θ_i^* , and so it can be viewed as a *recognition function* $\rho_\Theta : \{\Theta_i^*\} \rightarrow \{\theta_i\}$, such that $\theta_i = \rho_\Theta(\Theta_i^*)$ is the value recognized for the reference property Θ_i^* .

$$\begin{array}{ccc} & \xrightarrow{\rho_\Theta} & \\ \{\Theta_i^*\} & & \{\theta_i\} \\ & \xleftarrow{\iota_\Theta} & \end{array}$$

The $\iota_\Theta - \rho_\Theta$ cycle is again crucial, since it constitutes, complementary to the $\iota_P - \rho_P$ cycle, a *link between the empirical realm and the informational realm* in the model of the property intended to be measured. Under the hypothesis that each element in $\{\Theta_i^*\}$ is stable, the triple $\langle \{\theta_i\}, \{\Theta_i^*\}, \iota_\Theta \rangle$ constitutes an instrument-independent, and therefore in principle *public*, scale.

Definition 5. (Public scale module): **public-scale** $\stackrel{def}{=} \langle \{\theta_i\}, \{\Theta_i^*\}, \iota_\Theta \rangle$

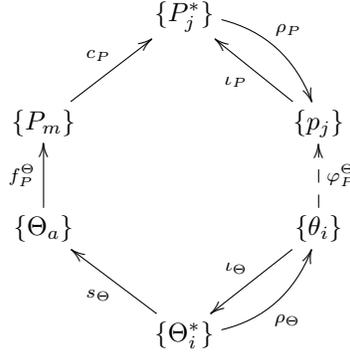
Once a scale is defined and made publicly available, the problem of linking it with the private scale of the instrument, as made possible by the transduction module of the instrument, arises. The operation that solves the problem, by turning *private* pre-measurement into *public* measurement, is the instrument calibration.

5.2 Instrument calibration: calibration module

When both the private scale and the public scale are available, the transduction f_P^Θ is exploited for identifying their relation: this is the goal of instrument calibration. The instrument is put in interaction with objects whose temperature is an element of $\{\Theta_i^*\}$, and the process is followed exactly as in the case of a pre-measurement, thus obtaining an element of $\{P_j^*\}$ and then an indication value p_j for each Θ_i^* . Differently from what happens in a pre-measurement, however, in this case the temperature value θ_i is also known of the temperature Θ_i^* of the object with which the instrument interacts. Hence for each temperature Θ_i^* a pair of values

$$\langle \theta_i, p_j \rangle = \langle \rho_{\Theta}(\Theta_i^*), \rho_P \circ c_P \circ f_P^{\Theta} \circ s_{\Theta}(\Theta_i^*) \rangle$$

i.e., (public identifier, private identifier), is obtained, where s_{Θ} is the function that maps each element of $\{\Theta_i^*\}$ into itself, intended as an element of $\{\Theta_a\}$, i.e., the canonical injection of $\{\Theta_i^*\}$ into $\{\Theta_a\}$. By repeating the process for all elements of $\{\Theta_i^*\}$, a set $\{\langle \theta_i, p_j \rangle\}$ of pairs is constructed, which is the extensional definition of the calibration function φ_P^{Θ} , $p_j = \varphi_P^{\Theta}(\theta_i)$, that is $\rho_P \circ c_P \circ f_P^{\Theta} \circ s_{\Theta}(\Theta_i^*) = \varphi_P^{\Theta} \circ \rho_{\Theta}(\Theta_i^*)$. Note that, according to [18, def. 4.31], a calibration curve is a “relation between indication and corresponding measured quantity value”, hence the inverse of φ_P^{Θ} . The rationale for calling φ_P^{Θ} “calibration function” is that it is this function that is constructed while calibrating the instrument, while, as shown below, its inverse is used in performing measurements by means of the thus calibrated instrument.



The identity $p_j = \varphi_P^{\Theta}(\theta_i)$ states the *empirical fact* that the indication value p_j corresponds to the reference value θ_i , where the key point here is that a relation between informational entities is derived from an empirical fact. We call the triple $\langle \{\theta_i\}, \{p_j\}, \varphi_P^{\Theta} \rangle$ a *calibration module*.

Definition 6. (Calibration module): $\langle \{\theta_i\}, \{p_j\}, \varphi_P^{\Theta} \rangle$

This highlights that calibration is the informational counterpart of transduction, so that the calibration function can be intended as the mathematical model of the transduction. By exploiting an instrument whose behavior has been modeled by φ_P^{Θ} , a measurement is expected to produce a temperature value θ_i from an indication value p_j , thus requiring φ_P^{Θ} to be invertible. A configured measuring instrument equipped with a calibration function obtained in reference to a public scale is a *calibrated instrument*, which can be used to measure temperatures and to produce public information.

Remark 6. In the case the property under measurement and the instrument indications are embedded in an algebraically sufficiently rich structure and the transduction is modeled so that φ_P^{Θ} is a parametric analytical function, the calibration could only require to obtain the pairs $\langle \theta_i, p_j \rangle$ for identifying the values of the parameters – e.g., two pairs in the case φ_P^{Θ} is linear.

The invertibility condition on φ_P^{Θ} is crucial for the instrument to be used to attribute values to temperatures, since this is possible only if the inverse $\varphi_{\Theta}^P = (\varphi_P^{\Theta})^{-1}$ of the calibration function φ_P^{Θ} can be constructed. In turn, this requires the selection of a public scale which is not finer than the private scale. Still, since both f_P^{Θ} and c_P are in principle many-to-one this is a delicate issue, also connected to the fact that measurement results are to be intended as involving uncertainty. In fact, if the composite $c_P \circ f_P^{\Theta}$ is many-to-one, then two reference temperatures can be mapped to the same indication, and so two reference values can be associated via φ_P^{Θ} to the same indication value. In order to avoid this, sets of reference values can be put in one-to-one correspondence with indication values, but this implies that the measurement result is to be expressed in terms of best reference value and uncertainty (a

more encompassing position is mentioned in [27, p.3758], where “an inverse, partial inverse, approximate inverse or pseudoinverse” of the calibration function is allowed).

6 Second model: ideal measurement

Given the sequence

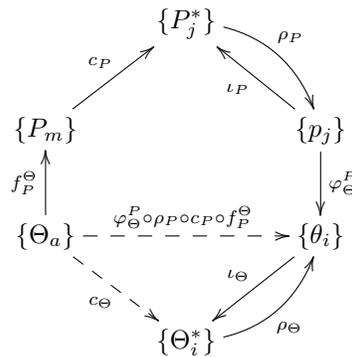
1. instrument configuration (including private scale construction),
2. pre-measurement,
3. instrument calibration (including public scale construction),

measurement can be characterized as a pre-measurement followed by the application of the calibration function. Since we are still not taking measurement uncertainty into account, we call this process an *ideal measurement*.

6.1 Ideal measurement: evaluation

The instrument is designed so as to be able to attribute a value to an input property. This is achieved by making the calibrated instrument interact with the property to be measured Θ_a , thus obtaining a *measured value* θ_i according to the following procedure:

1. via the transduction function f_P^Θ the temperature Θ_a is associated with a transduced position $P_m = f_P^\Theta(\Theta_a)$;
2. via the matching function c_P the transduced position P_m is associated with a reference position $P_j^* = c_P(P_m) = c_P \circ f_P^\Theta(\Theta_a)$ in the private scale, i.e., an instrument indication;
3. via the recognition function ρ_P the indication P_j^* is associated with an instrument-specific value $p_j = \rho_P(P_j^*) = \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$, i.e., an indication value;
4. via the inverse of the calibration function, $\varphi_\Theta^P = (\varphi_P^\Theta)^{-1}$, the instrument-specific value p_j is associated with an instrument-independent value $\theta_i = \varphi_\Theta^P(p_j) = \varphi_\Theta^P \circ \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$, i.e., a value of temperature.



The relation $\theta_i = \varphi_\Theta^P \circ \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$ states the *empirical and informational fact* that θ_i is the value that identifies Θ_a in the *public scale*. We call the final triple $\langle \{\Theta_a\}, \{\theta_i\}, \varphi_\Theta^P \circ \rho_P \circ c_P \circ f_P^\Theta \rangle$ the *measurement module*.

Definition 7. (Measurement module): $\langle \{\Theta_a\}, \{\theta_i\}, \varphi_\Theta^P \circ \rho_P \circ c_P \circ f_P^\Theta \rangle$

This is the module that allows us to obtain a public value for a property under measurement, a conclusion that is expressed by the following

Claim 2. the value of Θ_a is θ_i in **public-scale** if $\theta_i = \varphi_{\Theta}^P \circ \rho_P \circ c_P \circ f_P^{\Theta}(\Theta_a)$

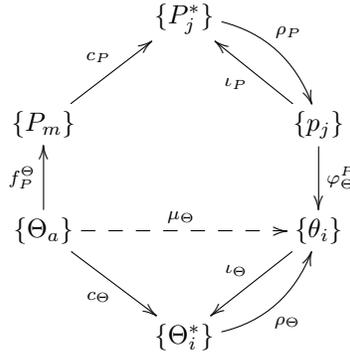
This is the public counterpart of *Claim 1*, with the difference that here the double implication does not hold, given that θ_i could be obtained as the value of Θ_a in **public-scale** by means of many different measuring instruments, and therefore independently of $\varphi_{\Theta}^P \circ \rho_P \circ c_P \circ f_P^{\Theta}$. Let us define the following *direct measurement* function.

Definition 8. $\mu_{\Theta} \stackrel{def}{=} \varphi_{\Theta}^P \circ \rho_P \circ c_P \circ f_P^{\Theta}$

Then we obtain $\mu_{\Theta}(\Theta_a) = \theta_i$, i.e., the value of Θ_a is θ_i in **public-scale**, which is usually written in the standard form

$$\Theta_a = \theta_i \text{ in public-scale}$$

In fact, μ_{Θ} is the black box model of the behavior of a calibrated measuring instrument: by opening the box the structure $\varphi_{\Theta}^P \circ \rho_P \circ c_P \circ f_P^{\Theta}$ is revealed. Furthermore, by exploiting the identification function ι_{Θ} , a *matching function* $c_{\Theta} : \{\Theta_a\} \rightarrow \{\Theta_i^*\}$ is obtained, so that $\Theta_i^* = c_{\Theta}(\Theta_a) = \iota_{\Theta} \circ \varphi_{\Theta}^P \circ \rho_P \circ c_P \circ f_P^{\Theta}(\Theta_a)$, stating the fact that through measurement the reference temperature corresponding to Θ_a has been identified to be Θ_i^* .



Thus the well-known claim is justified that *the core of a direct measurement process consists in a comparison between the property under measurement and the reference properties in a public scale.*

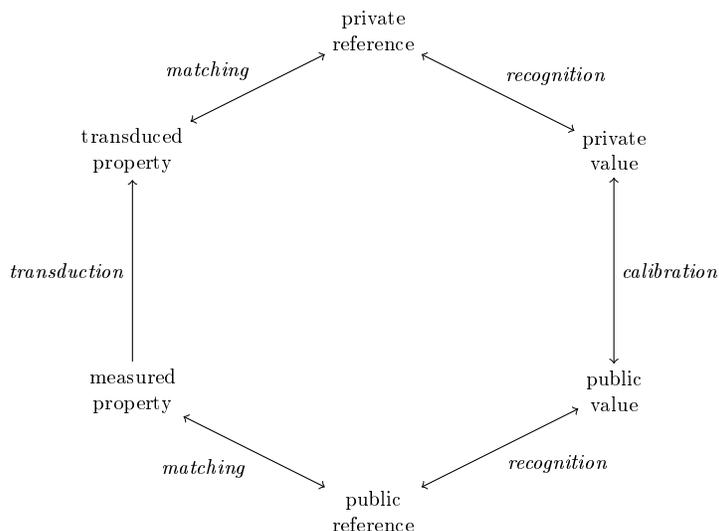
Remark 7. This model is sufficiently general so as to include as specific cases the measurements in which the property under measurement is directly compared with the reference properties of the public scale, as in the example of the measurement of mass by means of a two pan, equal arm balance. In these cases the structure of the process is simpler: once the public scale has been constructed, the empirical stage of the process consists in the matching $c_{\Theta} : \{\Theta_a\} \rightarrow \{\Theta_i^*\}$, so that the direct measurement function is defined as $\mu_{\Theta} \stackrel{def}{=} c_{\Theta} \circ \rho_{\Theta}$. Thus, the empirical work is entirely done by the recognition module for Θ .

6.2 The stages of the ideal measurement

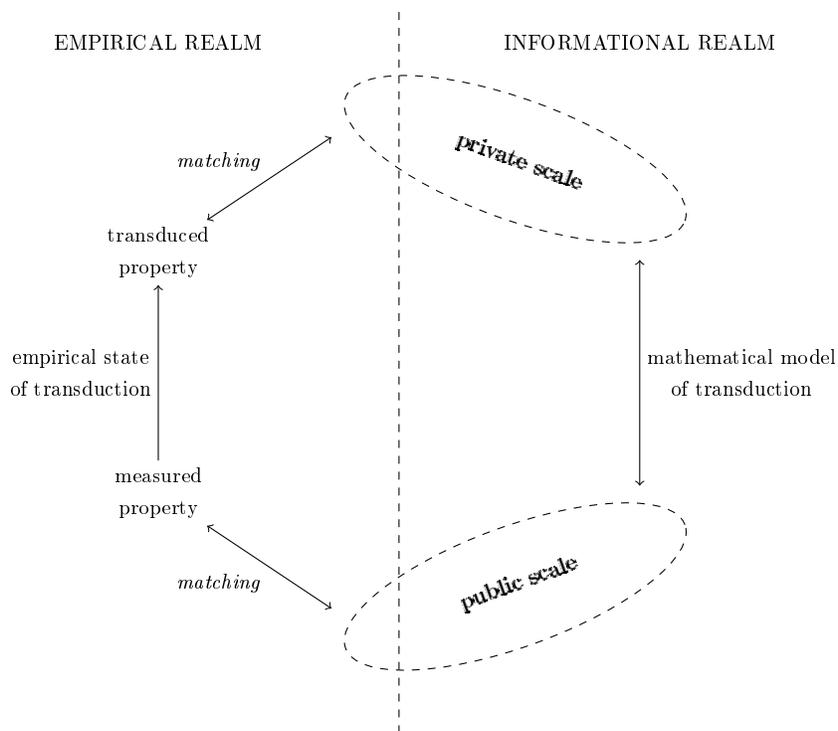
The previous analysis shows that measurement is a process performed through the stages 1-4 of pre-measurement, as in section 4.3, and:

5. the preliminary construction of a public scale module, which is partly *conventional* and partly *theoretical*, being based on the choice of the reference objects, the reference properties, and the reference values, and the assumption of the stability of the reference objects;
6. the application of a calibration module, which constitutes the model of the transduction and is computed at a different stage with respect to its definition, so that its validity depends on the hypothesis of the stability of the instrument, and therefore is *theoretical*.

According to this ideal model, a measurement is able to produce both *object-related*, i.e., objective, and *subject-independent*, i.e., intersubjective, *information*.



This is the outcome of a process whose structure has some significant symmetries.



As mentioned above, the functions we have considered model processes that are conceptually independent of each other, so that the entity modeled by a composed function only depends on the models of the component functions: hence only the non-composed functions have been considered in modeling measurement.

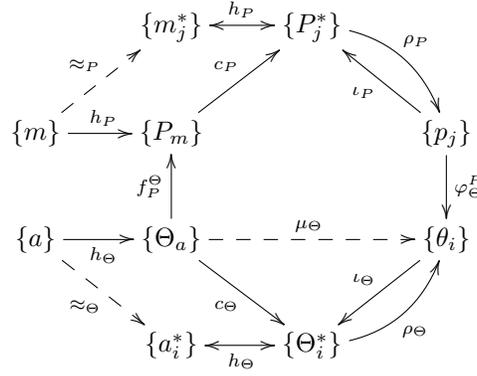
6.3 An extension: including objects in the model

While the entities taken into account so far are individual properties, i.e., properties of objects or values of properties, the model can be easily extended so as to include also the objects having the relevant properties, by introducing:

- a set $\{a\}$ of objects having temperature, a subset of which is $\{a_i^*\}$, the set of reference objects whose temperatures are in $\{\Theta_i^*\}$, chosen for setting the public scale; we call $\{a_i^*\}$ a *realization* of the public scale;
- a set $\{m\}$ of columns of mercury in the thermometer, a subset of which is $\{m_j^*\}$, the set of columns whose positions are in $\{P_j^*\}$, chosen for setting the private scale (note then the distinction between a column of mercury – an object – and the position of its upper surface – a property); we call $\{m_j^*\}$ a *realization* of the private scale.

On this basis the model is extended by including:

- the function $h_\Theta : \{a\} \rightarrow \{\Theta_a\}$, that associates objects that can interact with the thermometer to their temperatures;
- the function $h_P : \{m\} \rightarrow \{P_m\}$, that associates columns of mercury to their positions;
- the functions $\approx_\Theta : \{a\} \rightarrow \{a_i^*\}$ and $\approx_P : \{m\} \rightarrow \{m_j^*\}$, that are object-related analogous of the matching functions c_Θ and c_P respectively.



Since in a measurement process objects are compared with respect to their properties, this extension, though still ideal, increases the realism of the model, which remains purely structural as the dynamical character of the measurement process is not explicitly considered yet. This enriched structure enables us more effectively to compare the present model of measurement with models proposed in previous works.

7 Comparison with recent models in the literature

In the previous sections we have progressively built a structural model of an ideal direct measurement in terms of the properties involved in the process. In this section we compare the proposed model with some other models presented in the recent scientific literature. We will show that the model we have proposed constitutes both a generalization and a completion of these previous models.

7.1 Comparison with the representationalist models

The representationalist model assumes that measurement is a mapping from empirical objects to informational entities such that relevant relations between empirical objects are preserved in the mapping by relations between numerical entities (representational theories are in fact usually presented in terms of numerical entities, as for example in [28, p.54]: the generalization to informational entities makes the analysis clearer and more consistent). Thus, measurement consists in constructing and applying a *measurement scale*, i.e., a morphism from an empirical relational system to an informational relational system, so as to allow us to encode information about objects under

measurement in terms of numbers or other informational entities. Accordingly, measurement is based on a theory proving that some relations among empirical objects can be represented by some relations among informational entities, where a consistent representation requires to ascertain existence and uniqueness conditions on the construction of a measurement scale. In turn, these conditions are presented in terms of an existence theorem, stating that there exists a map from the relevant empirical relational system to an informational relational system, and a uniqueness theorem, identifying the class of transformation up to which the mapping is unique.

The representationalist model only captures the abstract structure of a measurement process [21], by assuming the existence of the matching module $\langle \{\Theta_a\}, \{\Theta_i^*\}, c_\Theta \rangle$ and focusing on the definition of the scale. It is then, essentially, a model of the measurand-related recognition function.

$$\begin{array}{ccc} & \xrightarrow{\rho_\Theta} & \\ \{\Theta_i^*\} & & \{\theta_i\} \\ & \xleftarrow{\iota_\Theta} & \end{array}$$

In particular, the focus is on the (possibly idealized) empirical conditions that the empirical structure on the set $\{\Theta_a\}$, extensionally interpreted as a set of equivalence classes of objects, is to satisfy in order to be represented by a numerical structure on the set $\{\theta_i\}$. To wit, if the set of values $\{\theta_i\}$ is part of an informational structure determined by an ordering relation, the representationalist model requires that some empirical procedures exist which allow us to compare the objects carrying properties in $\{\Theta_a\}$ in such a way that the axioms defining the order relation are satisfied. In this sense, the model is centered on the scale construction and the empirical conditions which ensure the possibility of such construction, and as such it abstracts from the role of measuring instruments, so that it applies indifferently to direct and indirect measurement. The present model of measurement is then a development of the representational viewpoint to the case of direct measurement.

7.2 Comparison with Giordani and Mari's model

In previous papers we proposed models of measurement with and without transduction. Specifically, in [12] the measurement process is decomposed into an *experimental component* consisting in the identification of the equivalence class which a given empirical object belongs to, or of the corresponding property, as typically obtained by comparison, and a *representational component*, consisting in the assignment of a value from an informational system to the determined equivalence class. This is a first improvement of the representational viewpoint, as the model integrates both the comparison module and the scale.

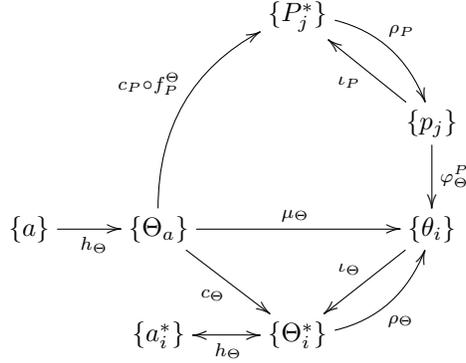
$$\begin{array}{ccccc} \{a\} & \xrightarrow{h_\Theta} & \{\Theta_a\} & \xrightarrow{\mu_\Theta} & \{\theta_i\} \\ & & \searrow c_\Theta & & \swarrow \iota_\Theta \\ & & \{\Theta_i^*\} & & \\ & \xleftarrow{h_\Theta} & \{\Theta_i^*\} & \xrightarrow{\rho_\Theta} & \end{array}$$

In addition, under the ideal condition that all properties in $\{\Theta_i^*\}$ are instantiated by the reference objects in $\{a_i^*\}$, so that the set of reference properties is in one-to-one correspondence with the standards that realize the definitions of such properties, the crucial notion of instrument calibration is imported into the picture.

In [13] the measurement process is further analyzed, in order to account for transduction, and decomposed into an *empirical stage*, where the interaction between the object and the instrument is accomplished and the property to be measured is related to an indication, and an *informational stage*, where the evaluation of the property is achieved and the indication value is related to a property value. Accordingly, obtaining a property value is a three step process:

1. a mapping from properties to indications, given by the composition of a transduction and a matching, $c_P \circ f_P^\Theta$;

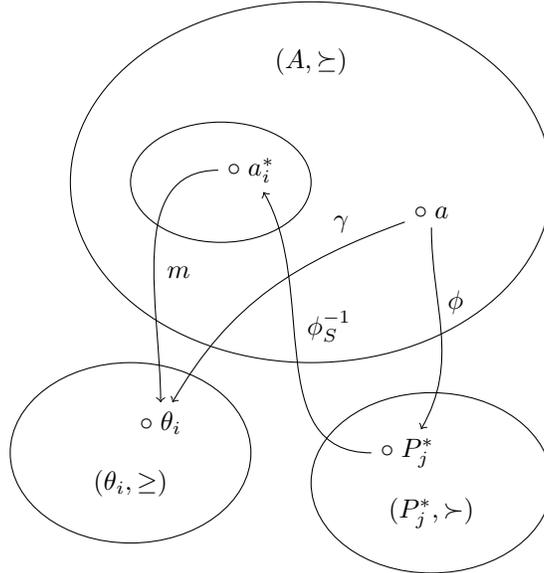
2. a mapping from indications to indication values, given by the recognition function ρ_P ;
3. a mapping from indication values to property values, given by the inverse φ_Θ^P of the calibration function.



Hence, the resulting value is precisely $\theta_i = \mu_\Theta(\Theta_a) = \varphi_\Theta^P \circ \rho_P \circ c_P \circ f_P^\Theta(\Theta_a)$. The present model is built on this analysis.

7.3 Comparison with Rossi and Crenna's model

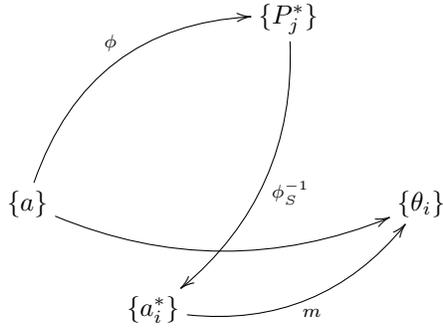
In a recent paper by Rossi and Crenna [31] the following model is considered (up to inessential changes of tags).



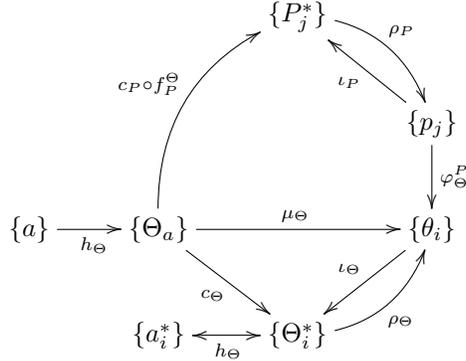
In this picture

1. ϕ maps objects to indications, so that $\phi = c_P \circ f_P^\Theta \circ h_\Theta$;
2. ϕ_S maps reference objects to indications, so that again $\phi_S = c_P \circ f_P^\Theta \circ h_\Theta$, where h_Θ is restricted to $\{a_i^*\}$;
3. m maps reference objects to values, so that $m = \rho_\Theta \circ h_\Theta$;
4. γ maps objects to values, so that $\gamma = m \circ \phi_S^{-1} \circ \phi$.

It is then not difficult to construe the schema along the following lines.



This can be then intended as a black box representation of the process, providing some basic conditions to be fulfilled. By opening the box (the need of opening the box is acknowledged also in [31], when it is stated that in modeling observation the primary interest is in considering the property Θ_a of the object as the input of the measurement process), its correspondence with our model is easily recovered.



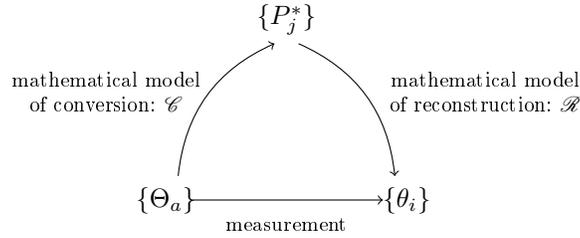
Of course, opening the box reveals the structure of the process and therefore provides a more explicit justification of its claimed trustworthiness. Furthermore, it is in some cases necessary if the process constraints are to be considered. For example, Rossi and Crenna’s model only works on the assumption that the set of the indications has the same cardinality as the set of reference objects, whose justification is in the fact that, for the calibration function to be invertible, $\{P_j^*\}$ has to be of the same cardinality as the set $\{\Theta_i^*\}$ of the reference properties. By contrast, it is not necessary for $\{\Theta_i^*\}$ to have the same cardinality as $\{\Theta_a\}$, since it is possible for both the transduction function and the matching function to be many to one.

7.4 Comparison with Morawski’s model

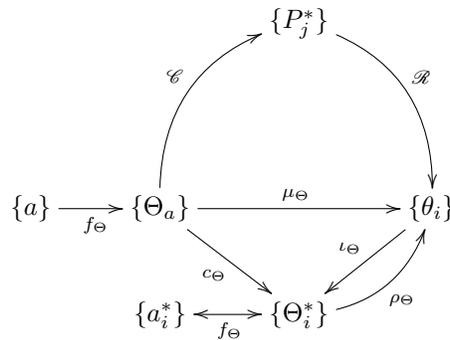
Finally, let us consider the model of measurement proposed by Morawski in [27] and primarily adapted to sensor-based instruments. This is based on mathematical models of (i) the measurand, (ii) the *conversion* process, mapping the measurand to the raw result of measurement, and (iii) the *reconstruction* process, mapping the raw result of measurement to the final result of measurement, which is the value attributed to the measurand. Morawski’s model is intended to capture both direct and indirect measurement. Since we are interested in the structure of direct measurement, the elements concerning the identification of the system under measurement, the definition of the measurand, and the computation related to the reconstruction of the measurand in an indirect measurement are abstracted away.

In its ideal version, where influence properties and uncertainties are not considered, the conversion is a map \mathcal{C} from $\{\Theta_a\}$ to $\{P_j^*\}$ that represents the transfer of information carried by the measurand into the domain of signals,

while the reconstruction is a map \mathcal{R} from $\{P_j^*\}$ to $\{\theta_i\}$ that represents everything that is required to establish the final result of measurement on the basis of the raw result, the mathematical model of conversion, and the available information on the measurand. Calibration is interpreted here as identification of the parameters of maps \mathcal{C} and \mathcal{R} .



Again, by opening the box some aspects of the model we are proposing become evident.



Hence, $\mathcal{C} = c_P \circ f_P^\Theta$ and $\mathcal{R} = \varphi_\Theta^P \circ \rho_P$. In addition, our model highlights that the reconstruction function allows us to move from the empirical realm of properties identified by address to the informational realm of properties identified by value, a feature that remains hidden if both \mathcal{C} and \mathcal{R} are modeled in terms of mathematical functions.

8 Third model: actual measurement

The model introduced so far is ideal, because the transduction has been assumed dependent only on the property to be measured, and the private scale and the public scale have been assumed to be stable. In this section we include some non-ideal conditions in the picture, such as influence properties and instabilities of the scales, and account for them in terms of *uncertainties* that affect the process and its results. This allows us to provide a more realistic interpretation of a direct measurement. In accordance with the choice of focusing on the structure of the process once the object under measurement is coupled with the measuring instrument in a specific environment, we do not consider here the problems related to the way the object and the environment are to be prepared for making the coupling possible in the expected conditions.

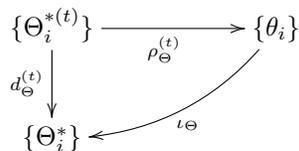
8.1 Measurement as a process affected by uncertainty

The result of an ideal measurement is the attribution of *one* value to the measurand, corresponding to the maximally specific information obtainable on the measurand given the adopted public scale (note that this characterization assumes measurement uncertainty to be relative to the public scale). In non-ideal conditions the measurement of a given property may lead instead to a multiplicity of values, possibly encoded as a probability distribution over the set $\{\theta_i\}$. Uncertainties in measurement can be classified in several ways (see, e.g., [16, clause 3.3.2] and [13, p.2149]): the modular structure of the model we are presenting provides a further, effective criterion of classification, if finally

interpreted in a *dynamic* context, in which the stages of the extended process of measurement are considered. In each stage some uncertainties may arise, and the uncertainty of the measurement result derives from their composition (how to combine uncertainty components may be a complex subject, also due to the fact that some components could be correlated, and the related literature is wide; it is a subject that we take for granted here: starting from [16] and [17], see, e.g. [20]).

As preliminary steps, we assume that a public scale for the measurand has been defined and a measuring instrument has been configured, so that in particular its private scale has been defined in turn. The following stages, that lead to produce a measurement result, develop according to a sequence that is only partially constrained.

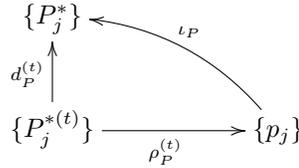
- *Measurand definition.* In the measurements usually performed in daily situations the measurand, i.e., the property intended to be measured and to which one or more values will be attributed [18, def. 2.1], is just the property with which the measuring instrument is made interact. This is an implicit condition of ideality, the same that we have assumed so far by using the same symbol Θ_a for the property applied to the transduction function f_P^Θ and the property applied to the direct measurement function μ_Θ . On the other hand, this definition makes the obtained information useful only in the here-and-now condition of the measurement, thus preventing a more general use of the reported information. For example, in measuring the temperature of a certain fluid the measurand would be the temperature of the portion of the fluid with which the thermometer interacted in the unknown conditions of the environment at the moment of the interaction. The measurand could be then defined *by description* in reference to given conditions of the object and the environment, for example under the assumption that the fluid is thermally homogeneous and at a given environmental pressure. This makes the information more transferable, at the price of a non-null definitional uncertainty, that takes into account the discrepancies between the conditions specified in the measurand definition and the actual conditions of the interaction with the measuring instrument (the measurand definition and its influence on the design and the operation of measurement is a subject that still requires investigation – see, e.g., [2] and [24]).
- *Dissemination of the public scale.* Given the definition of the reference properties Θ_i^* (thus in particular of the unit in the case of ratio quantities), in this stage the definition is realized in the reference objects a_i^* by means of which the measuring instrument will be calibrated. While the primary realizations derive from the *mises en pratique* (www.bipm.org/en/publications/mises-en-pratique), in the daily practice instruments are calibrated against working measurement standards, connected to the primary standards via a metrological traceability chain. Along the chain and across the time, inaccuracies and instabilities may affect the reproduction of the reference properties Θ_i^* , in such a way that the reference properties $\Theta_i^{*(t)}$ of the reference objects against which the instrument is calibrated at a given time t may differ from Θ_i^* . This requires us to parameterize the recognition function with respect to time t and to acknowledge that generally the resulting function $\rho_\Theta^{(t)}$ is only uncertainly known, given the uncertainty that affects the function $d_\Theta^{(t)}$ that models the discrepancy at t between $\Theta_i^{*(t)}$ and Θ_i^* .



- *Calibration.* At a given time t_1 the instrument is calibrated against some reference objects with reference properties $\Theta_i^{*(t_1)}$. This requires the instrument to interact with such reference objects, the reference properties $\Theta_i^{*(t_1)}$ to be transduced, and the instrument indications P_j^* produced by $\Theta_i^{*(t_1)}$ to be classified. Since both transduction and classification are empirical processes, influence properties and instrument instabilities may

affect the instrument indication. Hence both the transduction function and the classification function need to be parameterized with respect to the time t in which measurement is performed, thus obtaining a time dependent classification function $\rho_P^{(t)} \circ c_P^{(t)}$ and a conditioned transduction function $f_P^{\Theta, \mathbf{X}(t)}$, where $\mathbf{X}(t)$ is the set of influence properties that are known affecting the transduction. This dependence on time takes instabilities into account.

- *Transduction and classification.* The instrument is put in interaction with the object a at a time t_2 which is different from the calibration time t_1 . This implies that, generally, the instrument may behave at measurement time and at calibration time in different way, $f_P^{\Theta, \mathbf{X}(t_1)} \neq f_P^{\Theta, \mathbf{X}(t_2)}$ and $c_P^{(t_2)} \neq c_P^{(t_1)}$. As mentioned above, the dependence of these functions on time takes instabilities into account, and makes them only uncertainly known. Moreover, the empirical nature of the processes may make their results problematic, because the transduction could not stabilize to a fixed position and there could be errors in matching P_m with respect to $\{P_j^*\}$ (the so-called “reading errors” in the case of traditional instrument with analogue scales). Finally, also the private scale could be affected by instabilities, so that $\rho_P^{(t_2)} \neq \rho_P^{(t_1)}$, where the function $d_P^{(t)}$ that models the discrepancy at t between $P_i^{*(t)}$ and P_i^* is only uncertainly known.

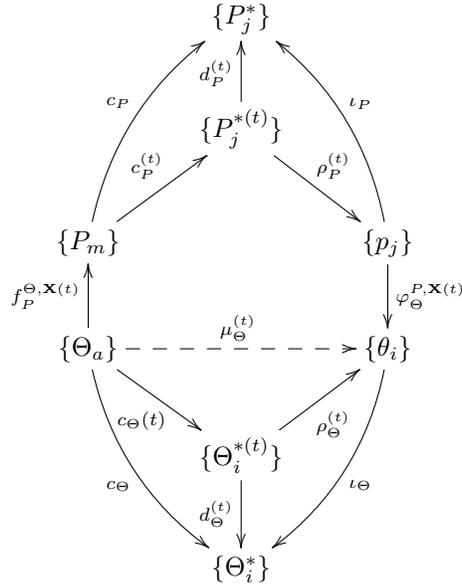


- *Measurand evaluation.* A value for the measurand is obtained by feeding the indication value into φ_{Θ}^P , which in the ideal case is just the inverse of the calibration function. Still, in this context some corrections can be introduced, that take into account the non-ideal conditions of the empirical stages of the process. For example, it could be acknowledged that temperatures and mercury positions are not exactly proportional, due to the fact the capillary in which the mercury flows is not perfectly cylindrical or due to second-order effects related to influence properties. Thus, the function φ_{Θ}^P is to be intended as a *reconstruction function* $\varphi_{\Theta}^{P, \mathbf{X}(t)}$, which also depends on the values assigned to the influence properties. While in the ideal case the measurement function is obtained by inverting the transduction function $f_P^{\Theta, \mathbf{X}(t)}$, where the only input property is Θ_a , in the actual case, $\varphi_{\Theta}^{P, \mathbf{X}(t)}$ is obtained by finding an inverse, a pseudo-inverse, or an inverse relative to a limited domain, of the transduction function $f_P^{\Theta, \mathbf{X}(t)}$.

This generalization of the model, once all the sources of uncertainty are taken into account, is sufficient for illustrating the main characteristics of an actual measurement and for identifying the sources of the uncertainty of the measurement result.

8.2 Actual structure of measurement

By taking these further elements into account, we obtain a more general model of actual measurement.



This diagram highlights in particular that the matching modules are more complex here than in the ideal case. In the case of the instrument matching, the comparison between observable properties P_m and reference indications P_j^* results from the composition of two functions, $c_P = d_P^{(t)} \circ c_P^{(t)}$, due to the fact that the actual matching is with reference indications $P_j^{*(t)}$, the associations with ideal reference indications P_j^* being computed via the discrepancy function $d_P^{(t)}$. An analogous complexity arises for the public scale matching. It is also worth noting that this analysis is consistent with the characterization of measurement uncertainty proposed by Sommer and Sieber in [33], where what they call the source unit can be viewed as carrying the uncertainty related to the standards in the calibration module, while the transformation unit and the indication unit can be viewed as dynamical versions of the transduction module and the calibration module of the present model. The main difference between these approaches consists in the fact that Sommer and Sieber based their study on a causal model of measurement, while the model proposed here is structural, a choice that makes it possible to introduce a comprehensive understanding of measurement which integrates the empirical, and so causal, aspect of the process with the informational aspect, that is crucial for obtaining measurement results. As just shown, this kind of model allows us to discuss in one framework the uncertainties derived from both causal interactions and representation.

8.3 Examples

Let us show how this model works in three other cases of direct measurement. Since we are interested in highlighting structural similarities, we present only the ideal models.

Example 1. Blood type measurement.

Blood type is a property of blood, related to the presence or absence of specific substances located on the surface of red blood cells. It is a nominal property, which allows us to categorize blood samples according to several different classification systems. Here we will consider the ABO system, which is based on a set $\{A, B, AB, O\}$ of four classes determined by the presence or absence of the A-antigen and B-antigen, where O is the class of red blood cells on which neither antigen is located. Antigens are found in human blood together with antibodies in such a way that the A-antigen is associated to the B-antibody and the B-antigen is associated to the A-antibody. **In addition, each antibody is able to agglutinate the corresponding antigen, thus generating reactions that can be observed. Since the presence of the A-antigen corresponds to the property P_A^* of being agglutinated in a certain kind of solution, the**

presence of the B-antigen corresponds to the property P_B^* of being agglutinated in another kind of solution, and so on, the various reactions allow us to measure blood type by placing a blood sample in solutions containing different kinds of antibodies. The entire process is captured by our model without difficulty. Let a be a blood sample.

1. Classification module:

- (a) $\{P_j^*\} = \{P_A^*, P_B^*, P_{AB}^*, P_O^*\}$, where P_j^* is the reference property of being agglutinated in a certain set of solutions;
- (b) $\{p_j\} = \{p_A, p_B, p_{AB}, p_O\}$;
- (c) ρ_P is the function such that $\rho_P(P_A^*) = p_A$, $\rho_P(P_B^*) = p_B$, and so on;
- (d) $\{P_m\}$ is the set of properties of being agglutinated in a certain set of solutions;
- (e) c_P is a comparison process relative to sets of states of agglutination.

2. Measurement module:

- (a) Θ is blood type, i.e., the property of having a specific antigen;
- (b) $\{\theta_i\} = \{A, B, AB, O\}$;
- (c) f_P^Θ is the agglutination effect process;
- (d) φ_Θ^P is such that $\varphi_\Theta^P(p_A) = A$, $\varphi_\Theta^P(p_B) = B$, and so on;
- (e) ρ_P and c_P are as above.

In this case, the calibration function φ_P^Θ is extensionally defined by (i) making a set $\{\Theta_i^*\}$ of reference standards, i.e. blood samples whose type is known, interact with the instrument and (ii) registering the relevant values in $\{p_j\}$ on the basis of the kind of observed properties in $\{P_A^*, P_B^*, P_{AB}^*, P_O^*\}$.

Example 2. Perceived quality measurement.

Students can be asked to evaluate the quality of a university course on a Likert scale, where each student is then part of the measuring instrument, together with a sheet presenting a question that explains what is intended to be evaluated and a printed sequence of 5 checkboxes, identified from *very bad* to *very good*. The measurement procedure specifies that each student has to check one and only one of such boxes.

1. Classification module:

- (a) $\{P_j^*\}$ is the sequence of checkboxes;
- (b) $\{p_j\} = \{1, \dots, 5\}$ is the set of ordinal numbers that identify each a checkbox;
- (c) ρ_P maps each checkbox to an ordinal number;
- (d) $\{P_m\}$ is the set of the possible answers that can be given to the question;
- (e) c_P is identity function, given that the transduction already maps to the reference properties.

2. Measurement module:

- (a) Θ is perceived quality;
- (b) $\{\theta_i\}$ is an ordered set of 5 values of quality, *very bad* to *very good*;
- (c) f_P^Θ is the process of checking one box;
- (d) φ_Θ^P maps each ordinal number that identifies a checkbox to a value of quality;
- (e) ρ_P and c_P are as above.

This is an example of the evaluation of a non-physical, ordinal property. With the aim of making the results produced by different students more meaningfully comparable, a list of exemplary descriptions of courses, each of them with an associated value of quality, could be made available to students. The list operates in the case as the public scale, against which the measuring instruments, i.e., the students, are calibrated by learning such prototypical cases of quality and their associated reference values.

Example 3. Light intensity measurement.

Light intensity can be measured by using photodetectors, which constitute basic components of photometers. Photodetectors can be characterized by the effect on which they are based to detect light. In particular, photodiodes convert light into electric current exploiting the photoelectric effect. In this case, the sensitive and substrate surfaces of photodiodes form a P-N junction operating as a photoelectric transducer that generates an electric current whose intensity, for monochromatic light, is proportional to the number of photons per unit area per unit time. Thus, comparing this current with a reference current allows us to determine the intensity of the light. Let a be some light radiation.

1. Classification module:

- (a) $\{P_j^*\}$ is a set of reference current intensities;
- (b) $\{p_j\} \subseteq \mathbb{R}[u]$ is a set of numerical values in a given unit u ;
- (c) ρ_P is defined so that the image of $\{P_j^*\}$ under ρ_P is $\{p_j\}$;
- (d) $\{P_m\}$ is the set of intensities of possible currents;
- (e) c_P is a comparison process of currents by intensity as provided by an ammeter.

2. Measurement module:

- (a) Θ is light intensity;
- (b) $\{\theta_i\} \subseteq \mathbb{R}[W/m^2]$ is the set of numerical values in unit W/m^2 ;
- (c) f_P^Θ is the photoelectric effect process;
- (d) φ_Θ^P is based on the quantum efficiency of the photodiode;
- (e) ρ_P and c_P are as above.

This example is interesting, as it highlights that the model we have proposed here captures the core process of a direct measuring instrument within a measurement process.

9 Conclusion

The structural model we have proposed in this paper is the result of taking into account and further developing in a general and consistent way the main elements that, according to the current literature, characterize a process of measurement performed according to a direct method, such that a measuring instrument is designed so as to directly interact, on the basis of a transduction effect, with the object under measurement relative to the measurand. One or more direct measurements are the core component of any measurement process, however complex it is, and this shows the importance of providing a general characterization of what direct measurement is. The bottom-up, modular structure adopted in the presentation is an effective strategy to handle the complexity of the process: each module is introduced in terms of the functional task it performs, first in the ideal, purely structural case and then in the real case in which the model becomes dynamic and measurement uncertainty has to be included in it. This reveals analytically the conditions that the process has to fulfill to guarantee the trustworthiness of its results. Such

a model is therefore a good basis for designing measurement processes whose results have an uncertainty not less than the definitional uncertainty, which characterizes the measurand, but not greater than the target uncertainty, as constrained by the goals of the process and the available resources to perform it.

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