Toward a harmonized treatment of nominal properties in metrology

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Abstract

This paper explores in a metrological perspective the basic characteristics of an (i) experimental process that (ii) provides publicly trustworthy information (iii) on the property of an object as a value of that property (iv) through the comparison of the property and a reference set of properties of the same kind, at the same time not requiring that the property is quantitative. The conclusion is that such a process, called here a *nominal property evaluation*, is not only both logically and operatively possible, but actually shares most of the fundamental features of measurement, and in particular the possibility to provide publicly trustworthy information. Hence the proposed conceptual framework paves the way toward a harmonized treatment of nominal properties in metrology.

Keywords: metrology; nominal property; nominal property evaluation; quantity vs quality in measurement.

1. Introduction

1.1 Measurement has been conceptually characterized by three basic features so far

Measurement is generally considered to have three basic features:

F1. being an experimental process,

F2. providing publicly trustworthy information, and

F3. giving quantitative information.

The first feature, F1, differentiates measurement from computational processes and thought experiments: that measurement is an experimental process is not controversial. According to the *International vocabulary of metrology* (VIM) [JCGM 2012], measurement is a "process of experimentally obtaining..." [def. 2.1], and in fact this establishes the sort of entity that measurement is.

F2 differentiates measurement from expression of (even expert) opinion and judgment based on (even long) experience: measurement includes protocols that allow everyone to evaluate not only the quantity under measurement, but also the reliability of the evaluation, usually in terms of measurement uncertainty.

Finally, in line with the Euclidean tradition, F3 assumes that quantities are specific properties and that only properties of which a quantitative structure is known can be measured, where the measurement result reports one or more values for the quantity intended to be measured relatively to a predefined quantity, taken as the unit. A more refined treatment would specialize quantities as, e.g., rational unitary quantities and differential unitary quantities [Dybkaer 2009], but such distinctions are not necessary here: the term "quantity" will be used to refer to the default case of additive properties with a unit, as usual for physical quantities.

1.2 Applying measurement to new disciplines

Grounding on features F1-F3 metrology, the "science of measurement and its application" [JCGM 2012, def. 2.2], evolved, thus justifying the epistemic prestige of measurement. This also explains why societal endeavors outside the acknowledged scope of measurement aim at such prestige, with the possible effect that the principles and methods of metrology are applied in new fields and the scope of metrology is widened in consequence. The complexity of this process is witnessed by the fuzzy threshold between measurements and non-measurements particularly "in the social sciences [where] most evaluations are not measure[ment]s, but rather mixtures of opinion and estimation" [Sawyer et al 2016, p. 384].

Such an evolution already happened in the history of measurement, with the progressive introduction of paradigms and techniques to make more and more properties measurable, as the thirty years of the history of

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the VIM show, with (clinical) chemistry, laboratory medicine, and biology incorporating the principles of metrology. That measurement is not only related to physics is accepted today, and further disciplinary extensions may be envisaged, in particular toward social sciences (the literature of social measurement is wide; two good entry points are [Duncan 1984] and [Michell 2004]; on the challenging subject of measurement across sciences, see, e.g., [IMEKO 2016]).

1.3 The third feature: measurement as quantification

Feature F3 – measurement as quantification – is sometimes just taken for granted. On this basis, the quantities of a given kind (lengths, masses, and so on) and numbers (natural, integer, rational, etc) are recognized to have the same structure: once a unit has been chosen, each quantity on an object is associated with a number, termed "numerical quantity value" in the VIM [JCGM 2012, def. 1.20], so that some relations and operations among numbers are supposed to apply to the related quantities too. For example, since $1+2=3$ then $1\,\mathrm{m}+2\,\mathrm{m}=3\,\mathrm{m}$, and if a and b are objects whose lengths, L_a and L_b, have been measured to be $1\,\mathrm{m}$ and $2\,\mathrm{m}$ respectively, then the length of the object c obtained by the linear juxtaposition of a and b thus becomes $L_c=L_a+L_b=3\,\mathrm{m}$ as outcome. Of course, the existence of such a structure-preserving function – termed "morphism": see [Krantz et al 1971] – is not a trivial, nor a purely formal, fact. It is known, for example, that kinds of quantity like Celsius temperature or mass density do not admit such an additive juxtaposition, thus requiring a different characterization of their being quantities.

The possibility of these numerical mappings paved the way to the discovery that different kinds of quantity can be mutually related in terms of numerical equations, and then to the development of systems of quantities ("set of [kinds of] quantities together with a set of non-contradictory equations relating [them]" [JCGM 2012, def. 1.3]). The benefits of a quantitative approach to the empirical world are manifest: via these equations hypotheses on experimental facts can be effectively validated, and explanations and predictions of experimental facts can be obtained. As Campbell wrote, "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science" [Campbell 1920, p. 267].

1.4 Exploring a new path to extend the application of metrological principles and methods

The three basic features of measurement F1-F3 are independent of each other: there are experimental but non-quantitative processes, quantitative but non-experimental processes, and so on. A new path for extending the application of metrological principles and methods can then be explored, by considering experimental processes that provide publicly trustworthy information in a metrological perspective, but are non-quantitative (an extended analysis of what we have presented as the third feature of measurement, and therefore of the role of quantification in measurement, is given in [Mari et al 2017], which can be intended as a conceptual background for the present paper). If length is a canonical example of a quantity, an example of a non-quantitative property is shape: even if the shapes S_a and S_b of the objects a and b can be combined (in the sense that two squares can make a rectangle), no numerical values can be generally associated to them such that the numerical value of the combination is the sum of the numerical values associated to S_a and S_b. Analogous conclusions can be drawn about multi-dimensional, e.g., vector or tensor, properties, which cannot generally be combined additively even when their components can (this case is analyzed by [Rossi 2014, ch. 7]).

In this scenario the principles and methods based on quantitative structures cannot be maintained to treat non-quantitative properties, and therefore the outcomes are weaker than in the usual, quantitative case. Nevertheless, non-quantitative properties might reveal socially useful and epistemologically interesting information. Some examples, proposed by the VIM [JCGM 2012, def. 1.30], are the sex of a human being, the color of a paint sample, the sequence of amino acids in a polypeptide. The VIM terms them "nominal properties", and according to the long tradition of the opposition quantity vs quality sometimes they are also called "qualitative properties" (see, e.g., [Pendrill, Petersson 2016]).

1.5 Purpose and structure of the exploration

Together with F1-F3, two further fundamental hypotheses are assumed on the nature of measurement:

H1. it is a process aimed at providing information as a value, or a set of values, of the quantity under consideration, i.e., a *quantity evaluation* for short;

H2. it is a process based on the comparison of the quantity under consideration and a reference quantity of the same kind, i.e., the unit.

The aim of the present paper is to explore in a metrological perspective the basic characteristics of a process that fulfills F1 and F2 and maintains the structural constraints implied in H1 and H2, i.e., a process that:
– is experimental (F1);
– provides publicly trustworthy information (F2);
– provides information on a property as a value of that property (H1);
– is based on the comparison of the property under consideration and a reference set of properties of the same kind (H2),
but generalizes F3, i.e., evaluates a nominal property. We will call it a *nominal property evaluation*.
The analysis of the parallelism between quantitative and non-quantitative evaluations is also inspired by the *Vocabulary of nominal properties and examinations* [Nordin et al 2010], which develops a terminology for non-quantitative evaluations in parallel to the VIM. The terminology of this Vocabulary is basically adopted here, with the exception of the term "evaluation", used instead of "examination" to designate the "process of experimentally obtaining one or more property values that can reasonably be attributed to a property" (a paraphrase of the VIM definition of 'measurement' [JCGM 2012, def. 2.1]): in this preliminary stage the term "evaluation" seems to better convey the generic sense of 'production of values'.
The present paper analyzes the conditions of "reasonable attribution" of values to properties.

The exploration proposed here has the following conceptual structure. In Section 2 a short, simplified version of the theory of property types (the meaning of "kind" and "type" is different here; see Sections 2.1 and 2.3), as originally proposed by Stevens [Stevens 1946], provides a framework in which quantitative and non-quantitative evaluation processes can be jointly considered and compared. In Section 3 it is shown that the evaluations of quantities and nominal properties share a common abstract structure (and therefore all conclusions apply also to ordinal properties, which, in a sense introduced in Section 2, are intermediate between quantities and nominal properties). Finally, in Section 4 it is discussed how most aspects of this abstract structure can be experimentally realized independently of whether the evaluation is quantitative or not, the only fundamental difference being in the construction of the reference set, the entity having the same structural role as the unit and of which the set of a unit and its multiples and submultiples is a specific case (see H2 above). This implies in particular that both metrological traceability and measurement uncertainty – two strategic tools to obtain publicly trustworthy information from measurement – have structural analogies for non-quantitative evaluations.
Whether the experimental process will be termed, e.g., "measurement", "qualitative measurement", "examination", or "evaluation", and whether this will be considered part of the scope of metrology, thus expanding it, or just improved by the lessons learned from metrology, is not of concern here.

2. Property and types of property

2.1 Property and kind of property
We know the empirical world by assuming the existence of objects with properties: an object (what the VIM calls a "phenomenon, body, or substance" [JCGM 2012, def. 1.1]) is an entity with an at least temporary identity and manifesting itself through interactions with its environment. Objects interact in multiple and different ways with their environment (a rod interacts with its environment through what we are used to consider its length, its mass, its color, etc): properties of objects are (related to) such modes of interaction, so that the length, mass, color, etc of the rod are some of its properties. According to this basic ontology, properties of objects can be classified by mutual comparability: the length of the rod *a* and the length of the table *b* are comparable in a way that is different from the way the length of the rod and the mass of the table are. This leads to the generalization that there exists an entity – length in the example – of which both the length of the rod and the length of the table are instances: it can be called a "kind of property". Two properties of objects are then instances of the same kind of property if they are mutually comparable.
Note that both kinds of quantity (e.g., length) and quantities of objects (e.g., the length of the rod *a*) are customarily termed "quantities", leaving the context to remove the ambiguity. Alternative terms for kinds of quantity and quantities of objects are, respectively, "quantities in a general sense" (as in the VIM2 [ISO 1993, def. 1.1 Note 1]), or "general quantities" for short, and "individual quantities". Exactly the same

applies for "property". In order to make our arguments clearer the explicit terms "general property" / "general quantity" and "individual property" / "individual quantity" will be used here whenever appropriate. The basic relation between the entities introduced so far is:

> the *general property* of an *object* is an *individual property*

e.g.:

> the length of the rod *a* is an individual length

This leads to formalizing the relation in functional terms, by understanding general properties as functions P whose arguments are objects o and whose values $P(o)$ are individual properties. Hence "the length of the rod a" can be written "length(rod a)" or "length$_{\text{rod }a}$" (see the examples in [JCGM 2012, def. 1.1, Note 1]). Henceforth it will be written P_i instead of $P(o_i)$ to simplify the notation and to reduce the risk of mistakenly considering general properties *to be* functions, instead of *to be formalized as* functions. Just as an example, a general property may be also intended as the set of all its instances, thus the range of the mentioned function.

2.2 Comparison of properties and its formalization

The comparability of (comparable) individual properties is a complex issue: two such properties might be not only different or equal, but also not noticeably different but not necessarily the same.

Similarity, or experimental indistinguishability, ~, is a particularly problematic relation, given that it is reflexive (any property is similar to itself: $P_i \sim P_i$) and symmetric (if two properties are similar the order in which they are considered is immaterial: $P_i \sim P_j$ if and only if $P_j \sim P_i$), but not transitive (given three properties, from the facts that the first and the second are similar and that the second and the third are similar, the conclusion that also the first and the third are similar does not follow: $P_i \sim P_j$ and $P_j \sim P_k$ do not imply that $P_i \sim P_k$).

The fact that comparability is generally non-transitive has the critical consequence that individual properties, and therefore also individual quantities, could not be represented in any sufficiently simple form. Indeed, the observation that P_i and P_j are indistinguishable would lead to represent them with the same symbol, and the observation that also P_j and P_k are indistinguishable would lead to represent also P_k with the same previous symbol; but since P_i and P_k could be distinguishable, this would lead to the situation in which distinguishable properties are represented by the same symbol, a case of information loss in representation. Moreover, since the comparison can be iterated, the outcome might be that all instances of a general property are represented by the same symbol, so that the representation conveys 0 bits of information, a paradox known as "sorites" (see [Hyde 2014]). This issue, which affects both nominal properties and quantities, does not seem to have a general solution better than assuming transitivity, and therefore modeling the comparison of individual properties as an equivalence relation. This is what it is accepted here (for an analysis of this issue see [Mari, Sartori 2007]).

2.3 Type of property

Under this assumption, all individual properties of a given kind can be compared via an equivalence relation, $\equiv$, such that for any two individual properties P_i and P_j of the same kind P, either $P_i \equiv P_j$ or non-$(P_i \equiv P_j)$. It is an empirical fact that in some cases non-equivalent individual properties can be compared also by order, $<$, so that if non-$(P_i \equiv P_j)$ then either $P_i < P_j$ or $P_j < P_i$ (this hypothesis of total and complete ordering does not apply, in particular, to vector properties: for the sake of simplicity we will consider here scalar properties). And it is also an empirical fact that in some cases individual properties can be additively composed, through an operation, $+$, such that $P_i + P_j$ is a new individual property that might be equivalent to one of an object o_k, $P_i + P_j \equiv P_k$. This additive composition is the conceptual basis of the Euclidean standpoint, as formalized in axiomatic terms by Holder [1901] and then, e.g., by Mundy [1987]. The basic idea is that if $P_i \equiv P_j$ then the property P_k such that $P_k \equiv P_i + P_j$ can be intended as $P_k \equiv 2P_i$, such that a third way of comparison between properties becomes possible, their ratio P_k / P_i, whose outcome is not Boolean but a number, in this case $P_k / P_i = 2$ (this is not the main subject of the present paper, and therefore the topic is only mentioned; the interested reader may find more information for example in the three volumes *Foundations of measurement* [Krantz et al 1971]).

In a seminal paper Stevens [1946] proposed a classification in terms of the conditions that a symbolic representation of properties – he called it a "scale" – has to maintain in order to preserve the available

information[1]. What follows is a simplified re-interpretation of the original way in which Stevens presented his argument on scale types.

Let $s(P_i)$ be the symbol by which the property P_i is represented. Then some possible cases are as follows.

C1. The information on equivalence is preserved whenever equivalent properties are represented by the same symbol and non-equivalent properties are represented by different symbols: if $P_i \equiv P_j$ then $s(P_i) = s(P_j)$, and if non-$(P_i \equiv P_j)$ then $s(P_i) \neq s(P_j)$. This implies that s is constrained to be an injective mapping, and analogously that any transformation τ of representation, $\tau(s(P_i))$, must also be injective: if $s(P_i) \neq s(P_j)$ then $\tau(s(P_i)) \neq \tau(s(P_j))$. As it has been seen, this seems to be a basic condition for informative representability as such.

C2. The information on order is preserved whenever an ordered set of symbols is used in representation and ordered properties are represented by analogously ordered symbols: if $P_i < P_j$ then $s(P_i) < s(P_j)$. This implies that s is constrained to be a strictly monotonic mapping, and analogously that any transformation of representation, $\tau(s(P_i))$, must also be strictly monotonic: if $s(P_i) < s(P_j)$ then $\tau(s(P_i)) < \tau(s(P_j))$.

C3. The information on ratio is preserved whenever the set of symbols used in representation has a multiplicative structure and the ratio of properties is represented by their numerical ratio: if $P_i / P_j = k$ then $s(P_i) / s(P_j) = k$. This implies that s is constrained to be a similarity mapping, and analogously that any transformation of representation must also be a similarity: if $s(P_i) = ks(P_j)$ then $\tau(s(P_i)) = k\tau(s(P_j))$. As already pointed out above, P is more specifically a rational unitary quantity [Dybkaer 2009].

Hence: (i) the general properties that satisfy C3 (and C2 and C1) are called "quantities"; (ii) the general properties that satisfy C2 (and C1, but not C3) are called "ordinal properties" (the VIM actually calls them "ordinal quantities" [JCGM 2012, def. 1.26]); (iii) the general properties that satisfy C1 (and neither C2 nor C3) are called "nominal properties"[2].

2.4 On the characterization of type of property

While Stevens' classification relies on conditions on symbolic representation of properties, an even more fundamental characterization is based on the idea that different procedures of comparison are possible for individual properties of the same kind – by equivalence, order, ratio, etc in the case of nominal properties, ordinal properties, quantities, etc respectively – under a condition of invariance of comparison. For example, a general property is ordinal, with respect to a given comparison procedure <, if its instances can be compared in such a way that the outcome, e.g., $P_i < P_j$ for two non-equivalent properties P_i and P_j, depends only on the compared instances P_i and P_j and the comparison procedure < but, in particular, not on the possibly chosen symbolic representations $s(P_i)$ and $s(P_j)$.

This shows that types are ordered in terms of the algebraic structure of their invariance conditions: quantities are algebraically richer than ordinal properties, because individual quantities can be compared not only by order but also by difference and ratio. Hence nominal is the weakest type, a conclusion in agreement with the

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1   Stevens's theory of "scales of measurement" is not without objections (for a synthesis of the criticisms see [Velleman, Wilkinson 1993]), also because Stevens himself did some questionable choices, such as calling "admissible" or "permissible" the scale transformations that are invariant, so that objecting under the principle that proscriptions inhibit research was easy. The basic idea behind a motto such as "good data analysis does not assume data types" is that there should be no type constraints in the mathematical treatment of data: by computing, say, the average of a sample of (numerically encoded) textual labels some interesting patterns in that sample could be discovered. On the other hand, our interest here is about measurement and measurement results, not data analysis: it is not that plausible that one would be encouraged to sum lengths and masses because maybe some interesting pattern could emerge...

2   This is an *exclusive* strategy of definition, such that quantities are not ordinal properties even though they fulfill C2, which is the condition characteristic of ordinal properties, and analogously ordinal properties are not nominal properties even though they fulfill C1, which is the condition characteristic of nominal properties. An *inclusive* strategy of definition assumes instead that (i) quantities satisfy C3 (and C2 and C1); (ii) ordinal properties satisfy C2 (and C1); (iii) nominal properties satisfy C1. According to this inclusive strategy, quantities are also ordinal properties and nominal properties, and ordinal properties are also nominal properties. Since our purpose here is to explore the metrological characterization of nominal properties, and not to define property types, we will adopt the conceptually simpler exclusive strategy: nominal are those properties that fulfill only C1.

fact that basically only classification-related procedures can be performed on instances of nominal properties.

An alternative approach to the characterization of types of properties is the one that has been adopted by the VIM so far. The first two editions of the VIM included only the definition of 'quantity', "attribute of a phenomenon, body or substance, which may be distinguished qualitatively and determined quantitatively" (basically the same phrasing is in both in the VIM1 [ISO 1984] and the VIM2 [ISO 1993]). This is not so useful, given that 'quantity' is defined in terms of 'quantitative determination'. The definition in the VIM3 can be intended as a tentative to provide a more specific and less circular definition: "property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference" [JCGM 2012, def. 1.1]. In the lexicon introduced here, this can be rephrased as "individual property that has a magnitude that can be expressed as a number and a reference". Moreover, since basically everything "can be expressed as a number and a reference", the actual content of the definition is "individual property that has a magnitude". This interpretation is confirmed by the definition that the VIM gives of 'nominal property', "property of a phenomenon, body, or substance, where the property has no magnitude" [JCGM 2012, def. 1.30], i.e., "individual property that has no magnitude" (for the sake of simplicity ordinal properties are not considered in this analysis). Hence, according to the VIM the distinction between a quantity and a nominal property is that only the former "has a magnitude" (for a discussion on the relation between 'quantity' and 'magnitude', see [Mari, Giordani 2012]).

2.5 Types of property and types of property evaluation

Characterizing property types on the basis on the way in which individual properties can be compared has an interesting consequence: it highlights that the type depends on the adopted comparison procedure, and therefore it is not an intrinsic feature of the general property under consideration. Hence the fundamental concept is not 'property type' but 'property comparison type' or – since in metrology comparison is instrumental to evaluation – 'property evaluation type': being nominal, or ordinal, etc (is surely a feature that individual properties inherit as cases of general properties, and) is not primarily a feature of general properties but of the way their instances are compared and then evaluated. Indeed, instances of the same general property can be compared by means of procedures related to different types (so that, e.g., diameters of spherical objects can be compared in purely ordinal way, by means of a sequence of sieves), and the type of a general property can be assumed as one related to the algebraically richest known comparison. This is in agreement with the historical development of knowledge. For example, temperature was first considered an ordinal property: new knowledge and better instruments and procedures led to finding ways to compare temperatures quantitatively [Chang 2007]. Assuming in the past temperature as an ordinal property was not a mistake, but just the effect of knowledge still to be refined (for a more extended analysis of the relations between types of properties and types of property comparisons / evaluations see [Giordani, Mari 2012]).

On this basis, our exploration switches now to the analysis of property evaluations, with the aim of assessing the differences between quantity evaluations – of which measurement is a case – and nominal property evaluations.

3. The common abstract structure of the evaluations of quantities and nominal properties

3.1 The basic framework

The rationale of proposing one conceptual framework embedding the evaluations of both quantities and nominal properties is the fact that the basic structure of such evaluations is the same. This commonality is shown in reference to two simple examples:

| the length of the rod *a* is 0.123 m | the shape of the rod *a* is cylinder |

They share the same abstract structure:

> *P* of *o* is *v*

or, as written above:

> P_o is *v*

where then a general property *P* (length or shape) of an object *o* (the rod *a*) has been evaluated and the result is reported as an entity *v* (0.123 m or cylinder).

These statements convey some information under the condition that more than one entity could appear in place of *v* (i.e., it is <the length of the rod *a* is 0.123 m> but it might have been <the length of the rod *a* is

0.234 m>; it is <the shape of the rod *a* is cylinder> but it might have been <the shape of the rod *a* is cone>). Hence it must be supposed that in both cases *v* belongs to a set V, that will be termed "reference set".

The usual formalization is indeed:

$$P_o = v$$

i.e.:

length$_{\text{rod } a}$ = 0.123 m	shape$_{\text{rod } a}$ = cylinder

where then the object $o := \text{rod } a$ can be intended as the argument of the functions $P := \text{length}$ and $P := \text{shape}$, whose values are $v := 0.123$ m and $v := \text{cylinder}$ respectively.

Given that length is a general quantity, this justifies the choice of calling 0.123 m a "quantity value" or "value of a quantity" (see [JCGM 2012, def. 1.19]), and cylinder a "property value" (or more specifically "nominal property value") (see also [Nordin et al 2010, def. 9]). Moreover, these are two cases of a process aimed at associating a (property) value to an individual property: such a process can then be called a "property evaluation".

3.2 Differences

This formalization also highlights an apparent difference between the two cases, related to the values that the evaluations can provide. The value $v := 0.123$ m implicitly includes, via the unit metre, the information on the reference set V from which 0.123 m is chosen, V being indeed the set of the, possibly non-integer, multiples of the metre. Hence, $v := 0.123$ m actually means $v := 0.123$ m in V, where the specification V remains implicit in most cases. On the other hand, the entity $v := \text{cylinder}$ does not include the information on the reference set V to which *v* belongs: it might be, e.g., $V_1 := \{\text{cylinder, other}\}$ or $V_2 := \{\text{cylinder, cone, sphere, cube, other}\}$, where $v := \text{cylinder}$ is more informative in the second case, $v := \text{cylinder}$ in V_2, than in the first one, $v := \text{cylinder}$ in V_1. Hence the information $v := \text{cylinder}$ is incomplete (in the functional formalization of *P* this is clear: $v := \text{cylinder}$ is not sufficient to specify the range of the function *P*, i.e., the set of its possible values, so that *P* as a function remains undefined). To solve the problem let us assume that a reference set V of shapes is given, of which *v* is an element, and complete $v := \text{cylinder}$ accordingly, $v := \text{cylinder}$ in V, which conveys the information that cylinder is a shape chosen in V. In order to make the two cases of evaluation actually comparable in their structure, this must be made explicit:

length$_{\text{rod } a}$ = 0.123 m (in V)	shape$_{\text{rod } a}$ = cylinder (in V)

This shows that quantity evaluations and nominal property evaluations share a common abstract structure.

3.3 More commonalities

An important point here is that both expressions "length$_{\text{rod } a}$ = 0.123 m" and "shape$_{\text{rod } a}$ = cylinder (in V)" convey an information of actual (and not formal) equality: they claim that the rod has a length that is equal – within the limits of the accepted experimental approximation – to the length obtained by multiplying 0.123 times the length conventionally defined as the metre, and has a shape that is equal – within the limits of the accepted experimental approximation – to the shape identified as a cylinder in the given set of shapes V (for a more extended analysis of the meaning of these equalities see [Mari, Giordani 2012]).

The consequence is that, differently from what the term unfortunately suggests, the values of nominal properties are not "names", a mistake made by several authors[3]. A length value, like 0.123 m, is not a

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3   There is a long history of misunderstandings on this subject. According to Stevens, for example, in the nominal case, which he exemplified in terms of "numbering of football players for the identification of the individuals", "the numerals are used only as labels or type numbers, and words or letters would serve as well" [Stevens 1946, p. 678]. It is correct that the values of a nominal evaluation do not have an algebraic structure, and therefore they are not numbers even if written as numerals. On the other hand, the example chosen by Stevens is misleading, as it was the one proposed by Campbell about hotel room "numbers" ("the fact that my room is 187 and yours is 58 does not mean that either or both of us have any nearer relation to the occupant of room 245 than to any other person; nor does it imply necessarily that there are 187 rooms in the hotel" [Campbell 1920, p. 269]). Of course, Campbell is supposing in this example that rooms are not numbered sequentially. Indeed, *these cases refer to the identification of objects, not to the evaluation of a property* (the same objection applies unfortunately also to the VIM, which proposes "ISO two-letter country code" as an example of a nominal property [JCGM 2012, def. 1.30 example 4]: it is indeed a code, not a property). That identifiers are not clearly distinct from their names is acceptable (even though Campbell oddly concluded that a numeral is "a black mark on a piece of paper or certain sounds which I utter", thus superposing the symbol and its physical realization, a further source of confusion), but surely this does not apply to the values of nominal evaluations: in our example, a shape value such as cylinder is clearly distinct

linguistic entity but a length: consider the difference between 0.123 m and "0.123 m", such that, e.g., "0.123 m" and "0,123 m" are different terms, in different linguistic contexts, for the same value. Exactly in the same sense, a shape value, like cylinder in V, is not a linguistic entity but a shape.

3.4 Nominal property evaluations and partitionings into equivalence classes

Some interesting issues arise by comparing a nominal property evaluation and a partitioning into equivalence classes. Let us consider the following two cases.

Case 1. A set of objects a_i is given, each having a shape $P(a_i)$, which can be somehow compared so to decide for each pair, a_i, a_j, whether $P(a_i) = P(a_j)$ or not. If this comparison process is repeated, and the hypothesis of transitivity is assumed, a partition of $\{a_i\}$ as a set of classes of shape is obtained as the outcome, and no values of properties are involved: what is obtained is whether any two objects of the set have the same shape or not, but what shape, thus their shape value, remains unknown.

Case 2. A reference set V for the property is given, thus listing the possible shapes, e.g., V := {cylinder, cone, sphere, cube, other}. The elements of V can then be realized by objects s_i – they might be called *evaluation standards* for the sake of generality – so that $P(s_1) = $ cylinder, $P(s_2) = $ cone, and so on. An evaluation of the shape of a candidate object a is then performed by comparing the shape of a, $P(a)$, with the shapes of the evaluation standards, $P(s_i)$, until the standard s is found that $P(a) = P(s)$, thus leading to attribute a value to the shape of a according to the inference that, e.g., if $P(s_1) = $ cylinder and $P(a) = P(s_1)$ then $P(a) = $ cylinder.

The distinction between these two cases is not specific to nominal property evaluations, as it can be seen in the example of a two pan balance: Case 1 corresponds to comparing objects of unknown mass and coming to the conclusion whether they have the same mass or not, but still lacking the information on the values of such masses; Case 2 corresponds to comparing an object of unknown mass and mass standards whose value is assumed to be known, so that a value of mass is finally attributed to the candidate object.

Hence, Case 1 produces a partitioning into equivalence classes and Case 2 produces a nominal property evaluation (or possibly a measurement), from which a partitioning into equivalence classes is immediately obtained, according to the rule that two individual properties / quantities belong to the same equivalence class if (and only if) their value is the same. This is analogous with the distinction between unsupervised and supervised methods of machine learning, for exampled for pattern clustering by means of a neural network: "Two main learning strategies can be adopted. If the target output values [...] are known [...], a supervised learning strategy can be applied. In supervised learning the network's answer to each input pattern is directly compared with the known desired answer [...]. In other cases, the target answer of the network is unknown. Thus the unsupervised learning strategy teaches the network to discover by itself correlations and similarities among the input patterns [...] and, based on that, to group them in different clusters." [Berthold, Hand 2007, p. 273].

In summary, partitionings into equivalence classes and nominal property evaluations are not identical processes: since partitionings do not imply evaluations but evaluations induce partitionings, partitionings may be intended as components of nominal evaluations (and of measurements), but partitionings as such are not nominal evaluations.

3.5 Uncertainty in nominal property evaluations

According to the *Guide to the expression of uncertainty in measurement* (GUM) [JCGM 2008], which is focused on the treatment of quantities, a measurement should generally convey information not only on the estimated value, v (the VIM calls it "measured quantity value" [JCGM 2012, def. 2.10]), but also on the standard uncertainty, u_v. The GUM shows that, under given conditions, v can be intended as the mean value of a probability distribution and u_v as the standard deviation of the mean, i.e., a location and a scale parameter respectively [ISO 2006]. In this perspective the formalization presented above for generic evaluations provides a property value, $v = P(o)$, that may be intended to correspond to the estimated quantity value, but lacks an uncertainty. On the other hand, measurement uncertainty has the crucial role of establishing the quality of the information obtained by measurement and therefore of providing information on the degree of public trust that can be attributed to measurement results: the common structure of the evaluations of quantities and nominal properties highlighted so far would then be seriously flawed without some uncertainty treatment for nominal property evaluations, that in turn would convey information on the quality of such evaluations, as seen below.

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from its name, which might be the English "cylinder", the Italian "cilindro", ... Hence it should be clear that the values of nominal evaluations are not names.

The GUM framework, based on parametric distributions, does not apply to nominal property evaluations (think about averaging two shapes, such as cylinder and cube...), but this is in fact only one of its acknowledged limitations (e.g., a single value is not appropriate to convey information on measurement uncertainty if the underlying distribution is strongly asymmetric). The GUM was then generalized by its Supplement 1 [JCGM 2008b], under the assumption of encoding the information on a quantity by a probability distribution as such.

This formalization based on distributions is not in principle constrained by the type of the property evaluation and in fact simply generalizes to nominal property evaluations. Let us consider again an evaluation $P_o = v$, where v belongs to a reference set V, e.g., V := {cylinder, cone, sphere, cube, other}. Let then F be a set of probability distributions over V, i.e., F := {f: V → R^+ such that $\Sigma_V f(v) = 1$}, where in the case of nominal evaluations the reference set V is finite and therefore f is a probability mass function. Exactly along the line of the Supplement 1, the evaluation can be generalized by assuming as value for P_o not an element of V but a whole distribution of such elements, i.e., $P_o = f$, where f belongs to F. For example (let us write $f(v) = y$ as (v, y) for simplicity), an element of F is f = {(cylinder, 0.7), (cone, 0.2), (sphere, 0.0), (cube, 0.0), (other, 0.1)}, so that $Prob$(rod a) = f conveys the information that the shape of the rod is cylinder with probability 0.7, cone with probability 0.2, and other with probability 0.1. The distribution f may be reported as the result of the evaluation, which now encodes information also on uncertainty, where the extreme case is the one of null uncertainty, for singleton distributions, where one element of V has probability 1 and all other elements have probability 0.

Even though a distribution on nominal property values is non-parametric and its moments are not defined, a pair of values may be obtained from it with a role analogous to a location and a scale parameter. For a nominal property the concept itself of location is not defined, and in fact of a distribution such as f a mean value cannot be computed. On the other hand, the mode of the distribution could be intended as its most representative value, such as v = cylinder in the previous example, given that f(cylinder) = 0.7 is the maximum probability in the distribution. A problem of this choice is that the mode is not necessarily unique, so that nominal evaluations whose result is a multi-modal distribution would not have a single representative location. Nominal properties are algebraically weaker than quantities, so that losing some of the mathematically rich tools of quantitative treatment is practically unavoidable. The analogue of a scale parameter for the distribution f is, for example, Shannon entropy $H(f) := -\Sigma_V f(v) \log_2 f(v)$, which can be indeed computed independently of the algebraic structure of the set V and therefore also for nominal properties. $H(f)$ is 0 for singleton distributions and maximum for uniform distributions, thus correctly conveying some information on the uncertainty of the evaluation whose results is the distribution f. For additional information and perspectives on the treatment of uncertainty in nominal property evaluations see, e.g., [Ellison et al 1998], [Possolo 2014], [Possolo, Iyerb 2017].

4. The common concrete structure of the evaluations of quantities and nominal properties

4.1 Measurement as evaluation that provides publicly trustworthy information

While all measurements are evaluations, not all evaluations are measurements: the framework introduced in the previous Section, that encompasses the evaluations of both quantities and nominal properties, is not sufficient to identify measurement as a specific kind of evaluation. In reference to the features F1-F3 proposed in Section 1.1, also adding the conditions about being experimental and giving quantitative information is not sufficient: not all experimental quantitative evaluations are measurements. What is missing is the remaining feature: *measurement as an evaluation that provides publicly trustworthy information*.

Such a feature has been characterized in terms of the object-relatedness and the subject-independence, "objectivity" and "intersubjectivity" respectively for short, of the information conveyed by measurement [Mari, Carbone, Petri 2012]: let us review these features and how the structure of a measurement process is able to embed them. We will finally discuss whether and how objectivity and intersubjectivity may apply also to nominal property evaluations.

4.2 Objectivity and intersubjectivity in measurement

Measurement is expected to convey information about the quantity object of the measurement, i.e., the measurand – the "quantity intended to be measured" [JCGM 2012, def. 2.3] – and nothing else. Measuring

systems are designed, set up, and operated to fulfill this condition of object-relatedness, and ideally they should behave as perfect filters which discard the effects of everything that is not the measurand and whose output depends then only on the measurand. The non-ideality of measuring systems is revealed by their inability to guarantee such a complete selectivity, and then by their being sensitive not only to the measurand but also to other quantities that the VIM terms "influence quantities" [JCGM 2012, def. 2.52]. Since the information produced by measurement is supposed to be usable independently of the measuring system by which it was obtained, the issue arises of characterizing the measuring system behavior in a sufficiently specific way so as to make it possible to extract the information on the measurand by filtering out the spurious information ("noise") generated by influence quantities. Instrumental uncertainty is a tool to quantify objectivity.

Measurement is expected to convey information that can be interpreted in the same way by different persons in different places and times. This requires that the information is reported in a way that is independent of the specific context and only refers to universally accessible entities, so that in principle its meaning can be unambiguously reconstructed by anyone. Metrological systems, including quantity units realized in measurement standards disseminated through traceability chains, are developed and maintained to fulfill this requirement. The appropriate calibration of the measuring system guarantees the metrological traceability of the information it produces, and therefore the condition of intersubjectivity. Calibration uncertainty, which includes all uncertainties related to the definition of the unit and its realizations in all measurement standards in the traceability chain, is a tool to quantify intersubjectivity.

Through objectivity and intersubjectivity of its results, measurement is a source of public trust: this is not because we know *that* we can rely on the information it produces – bad measurements are possible and are measurements nevertheless –, but because we know *how much* we can rely on it.

4.3 The concrete structure of measurement

A measurement result such as:

$\text{length}_{\text{rod } a} = 0.123(2) \, \text{m}$

(where 0.123 m is the measured value and 0.002 m is the standard uncertainty) is the outcome of a process in which the experimental application of the measuring system to the object under measurement (the rod *a*) is only one step: it is the structure of the whole process that embeds objectivity and intersubjectivity.

The starting point is the condition that the information on the measurand is reported in relational form relatively to a predefined reference, that for quantities is the unit. The measurement result means indeed that (i) the individual quantity $\text{length}_{\text{rod } a}$ and the individual quantity metre have been compared by ratio and that (ii) this ratio of lengths is equal to 0.123 with an uncertainty of 0.002. A fundamental problem of measurement is then how to compare the measurand and the unit, and to guarantee that the objectivity and intersubjectivity of this comparison are sufficient for the purpose of measurement. Let us schematically review the dynamic structure of a measurement process in this perspective. Of course, what follows is only aimed at discussing how the structure of measurement embeds objectivity and intersubjectivity, not at presenting a complete characterization of what measurement is (an extensive analysis of the structure of a measurement process is presented, e.g., in [Rossi 2014]).

STEP 1, *unit definition*:

$\text{unit} := Q(s_0)$

where s_0 is the object or the class of objects having the quantity Q defined as the unit (e.g., $\text{m} := \text{length}(s_0)$ being s_0 the path traveled by light in vacuum during a time interval of $1/299\,792\,458$ of a second). This implies that:

$Q(s_0) = 1 \, \text{unit}$

with no uncertainty.

STEP 2, *measurement standard realization / mise en pratique*:
the primary measurement standard s_1 is identified or built such that:

$Q(s_1) = Q(s_0)$ with an uncertainty u_1

so that $Q(s_0) = 1 \, \text{unit}$ with an uncertainty u_1.

STEP 3, *measurement standard dissemination*:
a sequence of measurement standards $s_2, ..., s_m$ is identified or built such that:

$Q(s_{i+1}) = Q(s_i)$ with an uncertainty u_{i+1}

where in the operation the standard s_{i+1} is calibrated against the standard s_i and each calibration generally introduces a contribution of uncertainty, so that u_{i+1} is a non-decreasing monotonic function of u_i and $Q(s_m) = 1$ unit with an uncertainty u_m.

STEP 4, *measurand definition*:
the measurand $Q(o)$ is defined, where o is the object under measurement, of which the measurand is then a quantity (e.g., the measurand is the length of the rod a, defined as the distance of the supposedly parallel surfaces limiting the rod at a given temperature). The presence of a non-null definitional uncertainty (e.g., due to the non perfectly parallel limiting surfaces) may be acknowledged.

STEP 5, *measuring system set up and calibration*:
a measuring system is set up so as to make possible its interaction with the object under measurement o with respect to the measurand $Q(o)$ (this applies to direct measurement method [JCGM 2012, def. 2.5]; in the case of indirect method the structure is more complex but the underlying logic is the same). The set up includes the calibration of the instrument against the working standard s_m, in order to identify the relation between possible measurand values and instrument indication values, and including the information on calibration uncertainty, which is not less than u_m. If the linearity of the relation is not guaranteed, several working standards s_{mj}, each realizing a different individual quantity, need to be used, and a calibration table / diagram is produced.

STEP 6, *interaction of the measuring system with the object under measurement*:
the measuring system is put in interaction with the object under measurement o with respect to the measurand $Q(o)$ and an indication is obtained. The presence of a non-null instrumental uncertainty may be acknowledged.

STEP 7, *obtainment of a measurement result*:
the information of instrument indication is applied to the calibration table / diagram and a measurement result is obtained for the measurand $Q(o)$, for example as a measurand value v and a standard uncertainty u_v, which cannot be less than the definitional uncertainty and depends on the calibration uncertainty and the instrumental uncertainty.

4.4 Objectivity and intersubjectivity as derived from the structure of measurement

The object-relatedness of the information produced by measurement depends on both the definition of the measurand (STEP 4) and the set up and operation of the measuring system (STEPS 5 and 6):
– in STEP 4 the quantity intended to be measured is identified, with a degree of specification formalized by the definitional uncertainty, the "component of measurement uncertainty resulting from the finite amount of detail in the definition of a measurand" [JCGM 2012, def. 2.27];
– in STEPS 5 and 6 the measuring system is set up and operated by putting it in interaction with the object under measurement. The critical feature for objectivity is then the selectivity of the instrument, "... such that the values of [the] measurand are independent of other measurands or other quantities in the phenomenon, body, or substance being investigated" [JCGM 2012, def. 4.13], as formalized by the instrumental uncertainty, the "component of measurement uncertainty arising from a measuring instrument or measuring system in use" [JCGM 2012, def. 4.24].
Hence, large definitional or instrumental uncertainties show a limited confidence in the possibility of attributing measurement results to an individual quantity, thus limiting objectivity.
The subject-independence of the information produced by measurement depends on the whole traceability chain (STEPS 1, 2, and 3), including the calibration of the measuring system (STEP 5):
– in STEPS 1, 2, and 3 the unit, i.e., the quantity to which the measurand is going to be compared, is defined and made reliably available through a metrological traceability chain [JCGM 2012, def. 2.42], a "sequence of measurement standards and calibrations that is used to relate a measurement result to a reference", by means of appropriate working measurement standards, "used routinely to calibrate or verify measuring instruments or measuring systems" [JCGM 2012, def. 5.7];
– in STEP 5 the available working standard is exploited to calibrate the measuring system, with the aim of guaranteeing that, in principle, the same measurand is associated with the same quantity value(s). This

underpins the social trust that any given quantity value means the same thing, i.e., operatively corresponds to the same quantity, for different individuals, in different places, at different times. The critical feature for intersubjectivity is then the metrological traceability, the "property of a measurement result whereby the result can be related to a reference through a documented unbroken chain of calibrations, each contributing to the measurement uncertainty" [JCGM 2012, def. 2.41], as formalized by the calibration uncertainty.

Hence, a large calibration uncertainty shows a limited confidence in the possibility of attributing the same measurement results to a given individual quantity, thus limiting intersubjectivity.

4.5 The concrete structure of nominal property evaluation: objectivity and intersubjectivity

The previous analysis highlights that the objectivity and the intersubjectivity of the information produced by measurements are features that derive from the structure of the whole metrological system. In this perspective let us review how STEPS 1-7 can be applied to the evaluation of a nominal property, again by referring to shape but now, in order to make the comparison more meaningful, considered as related to ink patterns whose shape evaluation is expected to lead to the identification of alphanumerical characters, as implemented in optical character recognition (OCR) software systems. This is indeed a good example of a nominal property, that ink patterns can have: with respect to the property character shape there is not a metric or an order among ink patterns: alphabetical order is conventional, and has nothing to do with character shapes, so that it cannot be operatively exploited in the character recognition (as in the case of Section 4.3, this presentation is simplified, being only aimed at discussing if and how also nominal property evaluations can produce objective and intersubjective information: specific measurement problems require expanding the description with more details).

STEP 1', *reference set definition*:
while the assumption that Q is a quantity guarantees that a whole reference set V can be generated from the unit by additive composition, i.e., in terms of the multiples and submultiples of the unit, in the case of a nominal property P the reference set V must be defined by explicitly identifying each element v_j in V such that:

$$v_j := P(s_{0,j})$$

where $s_{0,j}$ is an evaluation standard, i.e., the entity having the property P defined as the element v_j of the reference set. Hence, e.g., "a" := shape($s_{0,1}$), "b" := shape($s_{0,2}$), ... where $s_{0,1}$, $s_{0,2}$, ... may be prototypical ink patterns for "a", "b", ... or algorithms that procedurally describe the shapes (of course, in this case shape($s_{0,j}$) is to be intended as the shape generated by j-th algorithm, i.e., a glyph, not the "shape of the j-th algorithm"). This is an extensional definition: the shapes in V are chosen so as to be distinct and to cover all possible relevant cases of character shapes. There is no uncertainty in these definitions.

STEP 2', *evaluation standard realization*:
the primary evaluation standards $s_{1,j}$ are identified or built such that:

$$P(s_{1,j}) = P(s_{0,j}) \text{ with an uncertainty } u_{1,j}$$

(if $s_{0,j}$ is an algorithm, this step could be immaterial, and $u_{1,j}$ would be still null).

STEP 3', *evaluation standard dissemination*:
a sequence of evaluation standards $s_{2,j}$, ..., $s_{m,j}$ is identified or built such that:

$$P(s_{i+1,j}) = Q(s_{i,j}) \text{ with an uncertainty } u_{i+1,j}$$

where in the operation the standard $s_{i+1,j}$ is calibrated against the standard $s_{i,j}$ and each calibration generally introduces a contribution of uncertainty, so that $u_{i+1,j}$ is a non-decreasing monotonic function of $u_{i,j}$ (if $s_{0,j}$ is an algorithm, again this step could be immaterial, and $u_{i,j}$ would be null).

STEP 4', *evaluand definition*:
the evaluand, i.e., the property intended to be evaluated, $P(o)$ is defined, where o is the object under evaluation, of which the evaluand is then a property. The presence of a non-null definitional uncertainty may be acknowledged.

STEP 5', *evaluating system set up and calibration*:
an evaluating system (e.g., an optical scanner able to read ink patterns and produce corresponding matrices of pixels) is set up so as to make possible its interaction with the object under evaluation o with respect to the

evaluand $P(o)$. The setup includes the calibration of the system against the working standards $s_{m,j}$, in order to identify the relation between possible evaluand values and instrument indication values, and including the information on calibration uncertainty, which is not less than $u_{m,j}$. A calibration table is produced.

STEP 6', *interaction of the evaluating system with the object under evaluation*:
the evaluating system is put in interaction with the object under evaluation o with respect to the evaluand $P(o)$ and an indication is obtained (e.g., the ink pattern(s) is/are acquired by an optical scanner and the image is processed into a matrix of black-or-white pixels). The presence of a non-null instrumental uncertainty may be acknowledged.

STEP 7', *obtainment of an evaluation result*:
the information of instrument indication is applied to the calibration table and an evaluation result is obtained for the evaluand $P(o)$, for example as a probability mass function over V whose mode is chosen as the evaluand value v.

In synthesis, nominal evaluation systems (STEPS 1'-7') and measurement systems (STEPS 1-7) have many structural similarities, the only difference being in the way the reference set is defined (STEP 1) and then the evaluation / measurement standards are generated (STEP 2), disseminated (STEP 3), and finally exploited to calibrate evaluating / measuring system (STEP 5): while the additive structure of quantities allows us to define a single reference quantity, i.e., the unit, and from it to derive the whole metrological traceability chain, in the case of nominal properties the reference set needs to be defined extensionally, and the corresponding traceability chain must be maintained with one evaluation standard for each element of the set. The analysis proposed in Section 4.4 can be then repeated identically here: the objectivity and the intersubjectivity of the information produced by evaluations performed according to STEPS 1'-7' are features that derive from the structure of the whole system.

5. Conclusions

From the beginning of its study, the treatment of nominal properties in measurement-related contexts has been flawed by misunderstandings and mistakes, as exemplified by the already discussed case (see footnote 3) of "numbering of football players for the identification of the individuals", presented to justify that "the nominal scale represents the most unrestricted assignment of numerals" [Stevens 1946, p. 678]. The basic issue here is that there are no empirical properties involved and the assignment is not an experimental process, so that the analogy with a nominal property evaluation is only syntactical.
This conundrum can be interpreted in the light of the classical tripartition between syntax, semantics, and pragmatics (for a measurement-related introduction to the subject see [Mari 1999]):
– the numbering of football players for the identification of the individuals is a purely *syntactic* activity, which can be performed in a wrong way (e.g., if two players receive the same identifier) but is neither true nor false: being just identifiers, strictly speaking such numerals do not mean anything; it might be called a "nominal labeling";
– a nominal property evaluation is a nominal labeling involving a property of the object under evaluation, such that the value attributed to the property is supposed to bring some information on the property; this makes it a *semantic* activity, and in fact the results of such an evaluation are in principle either true or false; on the other hand, an unconstrained evaluation (as it could be obtained by a random assignment) is unable to provide information on the quality, and therefore the usefulness, of its own results;
– finally, a nominal property evaluation performed according to STEPS 1'-7' produces not only a property value but also some information on its uncertainty, thus enabling a *pragmatic* treatment of the evaluation result: this, and only this, may be considered an interesting generalization of a measurement.
Thanks to their analogous operative structure, measurement and nominal property evaluation share the two basic features of (i) being an experimental process and (ii) providing publicly trustworthy information, where their difference is only in the third feature: giving quantitative vs non-quantitative information. This seems to be a sufficient justification for pursuing a harmonized treatment of nominal properties in metrology.

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