QUANTITY AND QUANTITY VALUE [#]

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Abstract: The concept system around 'quantity' and 'quantity value' is fundamental for measurement science, but some very basic issues are still open on such concepts and their relation. This paper argues that quantity values are in fact individual quantities, and that a complementarity exists between measurands and quantity values. This proposal is grounded on the analysis of three basic "equality" relations: (i) between quantities, (ii) between quantity values, and (iii) between quantities and quantity values. A consistent characterization of such concepts is obtained, which is then generalized to 'property' and 'property value'. This analysis also throws some light on the elusive concept of magnitude.

Keywords: measurement science; quantity; quantity value

1. Introduction

Being the "process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity" [1, 2.1], measurement is an (or maybe even *the*) operative connector of quantities and quantity values. Hence the concept system around 'quantity' and 'quantity value' is fundamental for measurement science. For example, again according to the *International Vocabulary of Metrology – Basic and general concepts and associated terms* (VIM3) [1] measurement units and measurands are quantities, and true values and indications are quantity values. The relation between

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quantities and quantity values established by means of measurement is fundamentally characterized by measurement uncertainty, as alluded by the admission that more than one quantity value might be "reasonably attributed" to the quantity intended to be measured, i.e., the measurand.

While measurement uncertainty is a widely considered subject in measurement science, less discussed is the relation between 'quantity' and 'quantity value' as such, although some clues suggest that such relation is still open for better clarification. Just to remain in the context of JCGM documents, it has been written [2] that "in future editions of [the VIM] it is intended to make a clear distinction between the use of the term error as a quantity and as a quantity value. The same statement applies to the term indication. In the current document such a distinction is made. [On the other hand, the VIM3] does not distinguish explicitly between these uses". The reference is plausibly to some related inconsistencies in the VIM3, as when it is written that "indications, corrections and influence quantities can be input quantities in a measurement model" [1, 2.50 Note 2], thus in particular assuming indications as quantities, against the definition of 'indication' as "quantity value provided by a measuring instrument or a measuring system" [1, 4.1]. The solution proposed in [2] is to more sharply distinguish between quantities (e.g., 'indication') and quantity values (e.g., 'indication value'), whereas, on the contrary, sometimes such distinction is simply neglected, as when the simplification is assumed that "the input to the measurement system is the true value of the variable" [3] (it should be clear that in fact this is just a mistake: the input to a measurement system is a quantity, not a quantity value).

We argue that the issue is not only a lexical one, and rather that it highlights the need of a better understanding of such concepts and their relations. Taking into account quantities and their values is indeed the everyday practice in the scientific endeavor as soon as mathematical modeling and measurement come on the scene. Reducing the ambiguities about them – and some of the related concepts, in particular 'kind of quantity' and 'magnitude' – seems to be a worthwhile purpose.

The following discussion does not require measurement uncertainty to be kept into account: were our conclusions be accepted, they would be immediately generalized to the case where more than one quantity value is to be "reasonably attributed to a quantity".

The problem is analyzed in Section 2 and on this basis in Section 3 a solution is presented for characterizing the concepts 'quantity' and 'quantity value', and then generalized to 'property' and 'property value'. Section 4 is devoted to explore the delicate concept of magnitude and two alternative scenarios are proposed for its interpretation. The last Section synthesizes some highlights of the proposed framework.

2. The problem: what are quantities and quantity values?

Measurement is aimed at attributing quantity values to quantities [1]. Provided that this generic statement is accepted, the commonly acknowledged epistemic role of measurement implies that *establishing a relation between quantities and quantity values is a significant process*. It is not amazing then that the appropriate characterization of such concepts, and therefore of the related conditions of possibility of such relation, i.e., of measurability [4, 5], is considered a basic task for measurement science. On the other hand, both 'quantity' and 'quantity value' are not exempt from ambiguity, as excellently documented in [6]. The issue is further highlighted by the multiplicity of the lexical

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options. For these concepts, not always with a clear distinction between them, terms such as "magnitude" [7, 8, 9], "property" [10], "attribute" [11], "characteristic" [12], "manifestation" [13, 14], "quantity" [15], have been traditionally used.

2.1. 'Quantity'

The term "quantity" is unfortunately polysemic since it designates both entities such as length and entities such as the length of this object, i.e., sentences as "length is a quantity" and "the length of this object is a quantity" would be customarily accepted as correct. For the latter entity the three editions of the VIM have alternatively used "specific quantity" (VIM1, [16]), "particular quantity" (VIM2, [17]), and "individual quantity" [1], while never defining the concept as such. We will call *general quantity* an entity such as length, and *individual quantity* an entity such as the length of a given object. The basic conception will be assumed that:

an individual quantity is an instance of a general quantity

so that, e.g., the length of this object *is a* length. The concept of general quantity can be then interpreted in two complementary ways:

1. as a *kind of quantity*, i.e., as a respect of comparability, thus accounting for expressions such as "object ω_i and object ω_j are indistinguishable with respect to length Q" (or, more usually, "ω_i and ω_j have the same length"), which can be written as $\omega_i \approx_Q \omega_j$;

2. as a *function* mapping objects to individual quantities, thus accounting for expressions such as "the length of object ω", indeed customarily written in functional form as $Q(\omega)$, e.g, length(ω).

Interpretation 1 provides an explanation for the fact that individual quantities inherit the features of the kind they instantiate [18]. For example, the velocity of any object has dimension LT^{-1} because velocity has such dimension. Interpretation 2 highlights that any general quantity Q has a domain of objects ω to which it can be applied, and therefore that each of such objects ω has an individual quantity $q = Q(\omega)$ (general and individual quantities will be denoted by upper case and lower case characters respectively). Finally, 1 and 2 are straightforwardly connected by the condition that two objects are indistinguishable with respect to a general quantity Q when their individual quantities are the same: $\omega_i \approx_Q \omega_j$ if and only if $Q(\omega_i) = Q(\omega_j)$ (in Section 4 this concept of 'having the same quantity' will be further discussed).

General quantities and individual ones are related but significantly different entities: for example, while the concepts around 'system of quantities' (e.g., 'base quantity' and 'quantity dimension') and those related to quantity types (e.g., 'ordinal quantity') are defined in reference to general quantities, measurement is operatively concerned with individual quantities, such as measurement units and measurands. Indeed, the quantity to which "one or more quantity values [...] can reasonably be attributed" [1, 2.1] by means of measurement is an individual one, such as the length of this object, surely not a general one, such as length. Analogously a quantity unit is an individual quantity. For example, the meter is the length of a given phenomenon, i.e., the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.

(The term "quantity unit" is adopted here in place of the customary "measurement unit" because most of what follows applies to generic processes of value assignment – "evaluation" for short henceforth –, not only measurement, this lexical position being

consistent with specific usages such as "mass unit" as a short form of "unit for the quantity mass" [1, 1.9 Note 4]. The ambiguity between general and individual quantities is manifest in the coordinated terms "quantity unit" and "unit quantity": in the former the occurrence "quantity" refers to a general quantity, the unit for the considered general quantity, whereas in the latter it refers to an individual quantity, the individual quantity which is assumed as the unit.)

The basic formula of quantity calculus is then written [19]:

$$q = \{q\} \cdot [q] \tag{1}$$

where the left hand side term denotes an individual quantity, i.e., $q = Q(\omega)$ for a given object ω.

We will focus on individual quantities here, and therefore whenever the unqualified term "quantity" is used henceforth will be referred to an individual quantity.

2.2. 'Quantity value'

A note about terminology work is now appropriate. According to [20], an "intensional definition is a concise statement of what the concept is. It states the superordinate concept to [the defined] concept expressed by the designation and its delimiting characteristics". For example, in the mentioned definition of 'measurement' given by the VIM3 'process' is the superordinate concept (i.e., measurement *is a* process) and 'condition of experimentally obtaining...' is the delimiting characteristic. Analogously, the VIM3 assumes 'property' as the superordinate concept of 'quantity', plausibly for both general entities (length, being a general quantity, is a general property) and individual ones (the length of this object, being an individual quantity, is an individual property). We might expect that the same structure:

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defined concept: superordinate concept + delimiting characteristics

applies to 'quantity value'. On this matter the evolution of the definition of the concept 'quantity value' across the three editions of the VIM is a complex one:

- VIM1: "the expression of a quantity in terms of a number and an appropriate unit of measurement" [16, 1.17];

- VIM2: "magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number" [17, 1.18];

- VIM3: "number and reference together expressing magnitude of a quantity" [1, 1.19].

Even the fundamental nature of the entity is characterized differently in these definitions: while the VIM1 considers a quantity value to be a linguistic entity (an "expression"), according to the VIM2 it is a non-linguistic entity (a "magnitude"). As for the VIM3, the definition is phrased so that the superordinate concept is not stated ('number and reference' is hardly considered a single concept). Furthermore, the very concept of reference, adopted plausibly to generalize the one of measurement unit to the case of ordinal quantities, is undefined in the VIM3, and the nature of the entity is unfortunately problematic in its turn (according to [1, 1.19 Note 1] the reference in a quantity value can be either (i) a measurement unit, i.e., a quantity, or (ii) a reference to a measurement procedure, i.e., a reference to a description and therefore to a linguistic entity, or (iii) a reference material, i.e., a physical entity; a definition of 'reference' encompassing these options is hard to imagine).

The problem is further highlighted by considering again eq. (1), where $\{q\}\cdot[q]$ is a quantity value, according to the first option of the mentioned [1, 1.19 Note 1]. Maxwell

explained such equation as follows: "Every expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called Unit, and the number is called the Numerical Value of the quantity." [21].

By assuming that by "the name of a certain known quantity..." Maxwell meant what it could be generically called the expression of a quantity unit ("Measurement units are designated by conventionally assigned names and symbols" [1, 1.9 Note 1]), this sentence can be written in shorter form as:

$expr(q) = number, expr(unit)$

It is not clear why Maxwell chose to describe eq. (1) by focusing on *expressions instead of expressed entities* (analogously it is argued in [6, p.83]), and superposing numbers and numerals, i.e., their expressions (it is acknowledged that the distinction was not so well defined at that time; in [22] both "measurement is the assignment of numbers" and "measurement is the assignment of numerals" can be read, and numerals appear to be interpreted as both expressions of numbers and nominal values written as numbers).

This emphasis on expressions has generated some unfortunate misunderstandings since then. Discussing about expressions is sometimes actually useful, for example in search of standardization (consider the troubled story of the decimal separator, either the dot or the comma: an issue only related to expressions), and any quantity equation can be written, and more generally communicated, only by means of expressions denoting the involved entities. But it is most important to acknowledge that *a number does not coincide with its*

expression(s). For example, "2", "two", "10_{binary}", ... denote the same number but are different expressions, and an equation does not coincide with its written form: what is equated are the expressed entities, surely not their expressions. A significant case of the acknowledged importance of the distinction between expressions and expressed entities is the *Guide to the expression of uncertainty in measurement* (GUM) [23], that systematically and clearly writes about "evaluating and expressing" uncertainty, and in particular distinguishes between Type A and Type B evaluations, surely not expressions (from this point of view a better, even though longer, title would have been "Guide to the evaluation and expression of uncertainty in measurement": expression is required, but is only the very final and tiny step of the process described in the GUM).

Nevertheless, a confusing emphasis on expressions has remained in measurement, as witnessed by the following quotations (emphasis added):

- "the value of a quantity is generally *expressed* as the product of a number and a unit" [24];

- "the value of a quantity is its magnitude *expressed* as the product of a number and a unit" [25];

- [the value of a quantity is a] number and reference together *expressing* magnitude of a quantity" [1];

- [the value of a quantity is a] "set of a number and a reference constituting the quantitative *expression* of a quantity" [1, French version].

Even if the reference to expressions is removed from these definitions, their characterizations of the concept 'quantity value' remain non fully consistent with each other because of a significant superposition of (put in other terms: a not so clear

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distinction among) the three concepts 'quantity', 'quantity value', and 'magnitude of quantity'. It seems that the concepts '(individual) quantity' and 'quantity value' are worth some further clarifications.

3. The solution: addressed quantities and classifier quantities

Let us consider again Maxwell's eq. (1). It admits two complementary interpretations:

- a *theoretical* one, according to which the quantity q is stated to be equal to the multiple $\{q\}$ of the quantity unit $[q]$; hence, the symbol "·" designates here the iterated additive concatenation of replicated quantities, abstracting from the objects to which such quantities belongs;

- an *operational* one, according to which at least in principle there exists a procedure that allows constructing an object that is composed by properly concatenating $\{q\}$ replicas of the measurement standard that realizes the unit $[q]$, and the quantity q characterizes an object ω that is indistinguishable, relative to the general quantity Q, from this object, i.e., $q = Q(\omega)$; hence, in this case the symbol "·" designates the iterated additive concatenation of indistinguishable objects with respect to the given quantity.

The link between these two interpretations is constituted by the *assumption* that the quantity of the object composed of $\{q\}$ replicas of the standard realizing the unit $[q]$ is $\{q\} \cdot [q]$. Hence, if $Q(\underline{\omega})$ is the quantity of an object $\underline{\omega}$ that realizes the unit $[q]$ then in this case eq. (1) means:

$$Q(\omega) = Q(\underline{\omega}) \oplus Q(\underline{\omega}) \oplus ... \text{ repeated } \{q\} \text{ times} \tag{2}$$

where "⊕" designates the additive concatenation of quantities. Such concatenation operation is in principle applicable to any quantity, not only to those which realize units. For example, if for a given quantity $Q(\omega)$ and a given unit $[q]$:

$$Q(\omega) = 2 \cdot [q] = Q(\underline{\omega}) \oplus Q(\underline{\omega})$$

then an inference such as:

$$Q(\omega) \oplus Q(\omega) = 2 \cdot (2 \cdot [q]) = Q(\underline{\omega}) \oplus Q(\underline{\omega}) \oplus Q(\underline{\omega}) \oplus Q(\underline{\omega})$$

is correct. On the other hand, eq. (2) highlights that the term at the right hand side of the equation denotes a quantity in its turn, i.e., eq. (1) is in fact *an equation of quantities*, even though $\{q\} \cdot [q]$ is assumed to be a quantity value. This consideration is also confirmed by the inverse equation:

$$\{q\} = Q(\omega) \oslash [q]$$

(see, e.g., [1, 1.20 Note 2]), where the symbol "$\oslash$" designates here an operation that applies to quantities, further showing that measurands and quantity units are assumed to be entities that in some sense can be divided with each other, the result being a number.

3.1. Two theses

The previous considerations are the basis for two theses which synthesize the solution we propose to the problem under analysis.

Thesis 1: the entities involved in the left hand side and in the right hand side of eq. (1) are of the same sort; *both are individual quantities*.

This thesis is, in a sense, trivially included in the assumption that both measurands and quantity units are (individual) quantities, as the specific case shows in which $\omega \approx_Q \underline{\omega}$ (i.e.,

the object under consideration ω is shown as having the same quantity as the object ⍵ which realizes the unit) and therefore:

$$Q(\omega) = 1 \cdot [q] = Q(\underline{\omega})$$

Furthermore, the set of quantities under consideration is closed under the operation ⊕, i.e., both $Q(\omega) \oplus Q(\omega)$ and $Q(\underline{\omega}) \oplus Q(\underline{\omega}) = 2 \cdot [q]$ are quantities in their turn. Still, the fact that the same concept, 'individual quantity', includes both measurands and quantity units, which are distinctively different for several aspects, highlights its complexity. While in measurement measurands are assumed as empirically given unknown entities, quantity units are designed, and therefore known before measurement, in a process sometimes called *scale construction* which produces a structure of classifiers assumed to be adequate for the measurand under consideration. On this matter in [26] the metaphor of a net is proposed: given a territory whose points are the possible measurands, a net of regularly spaced meshes is constructed, each mesh corresponding to a (sub)multiple of the unit. On the territory the net is then drawn so that each point is included in one and only one mesh, i.e., each measurand is represented by one and only one (sub)multiple of the unit, and therefore, once the unit has been defined, by one and only one quantity value (the fact that we are not taking into account measurement uncertainty here is manifest). This complies with the condition that two operatively indistinguishable measurands are represented by the same quantity value, thus highlighting that *the set of quantity values generated by the quantity unit operates as a classification*, each quantity value being an element of it: each $\{q\} \cdot [q]$ is a distinguishable quantity, where $\{q\}$ identifies a class within the classification generated by $[q]$ through quantity concatenation.

By developing Thesis 1 two sorts of individual quantities can be singled out:

- the first sort includes quantities that are known by pointing out objects that instantiate them, e.g., the length $Q(\omega)$ of the object ω; here the object ω for the quantity $Q(\omega)$ can be suggestively called its *address*, so that $Q(\omega)$ may be called an *addressed quantity*, or *a-quantity* for short;

- the second sort includes quantities that are known as elements of a classification generated by a unit, e.g., $\{q\}\cdot[q] = 0.123$ m; hence, $\{q\}\cdot[q]$ may be called a *classifier quantity*, or *c-quantity* for short.

(We are aware that the terms "addressed quantity" and "classifier quantity" might appear cumbersome; we are just proposing them here for convenience, so to be able to reduce the ambiguities due to the lexical overloading mentioned above.)

Accordingly:

- measurands and quantities subject to measurement are a-quantities, i.e., individual quantities of objects which are considered as given and therefore can be indicated by address (such as radius of circle A, wavelength of the sodium D radiation, kinetic energy of particle i in a given system – examples from [1], Note 1 to the definition of 'quantity' – where the addresses are 'circle A', 'sodium D radiation', 'particle i in a given system' respectively);

- quantity values are c-quantities, i.e., individual quantities of objects which are constructed or constructible, in the simplest case by additive concatenation of objects which realize the quantity unit.

That a-quantities such as measurands and c-quantities such as quantity values are entities of the same sort, i.e., individual quantities, is pivotal for understanding the meaning of the equality in eq. (1), and this is further reasons why it is so important not to confuse

quantity values with the symbols by which they are expressed, as emphasized in Section 2.2. A quantity value is as real as a quantity subject to measurement: their difference is primarily in the way they are known, and only as a consequence in the way they are expressed. Indeed, and more generally, equality relations involving individual quantities are defined:

[*first kind of equality*; let us call it "α-equality", $=_\alpha$]: between a-quantities: for example, the length of this object can be ascertained to be equal to the length of that object, and this may be the outcome of an experimental process in which no information about quantity values is involved;

[*second kind of equality*; let us call it "β-equality", $=_\beta$]: between c-quantities: for example, it may be known that $2 \cdot 10^{-2}$ m is equal to 0.7874... in, an assessment which does not require any experimental activity once the involved quantity units have been defined and their relation identified.

The complementarity is manifest: the information of equality of a-quantities (i.e., α-equality) is empirical but not inter-subjective, since it supposes the access to the objects ω_i and ω_j to be useful; the information of equality of c-quantities, i.e., $\{q\} \cdot [q] =_\beta \{q\}' \cdot [q]'$, is inter-subjective, being universally shareable, but not empirical. Neither of them reports in fact a measurement result.

Equality relations are finally also defined:

[*third kind of equality*; let us call it "γ-equality", $=_\gamma$]: between a-quantities and c-quantities.

This is the interpretation of eq. (1) according to Thesis 1:

a-quantity $=_\gamma$ c-quantity

or also, equivalently:

individual quantity known by address $=_\gamma$ individual quantity known by value

We believe that the distinction between such three kinds of equalities, as also synthesized in Fig. 1, is crucial for the present discussion.
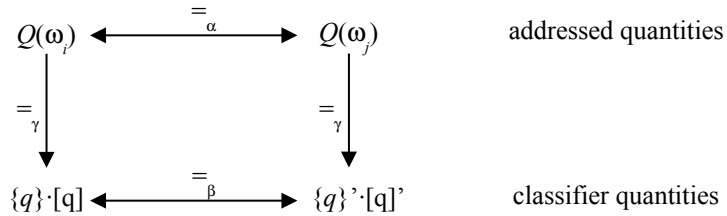

Figure 1 – The three kinds of equalities involved in the relations between a-quantities and c-quantities.

The very fact that three distinct equalities are possible between (addressed or classifier) individual quantities is a hint of the inherent complexity of our subject.

(The astute reader is surely not confused by a fourth case of equality in the analysis in Section 2.2, such that we concluded, e.g., that according to the VIM:

$v = number \cdot unit$

Indeed, this is just the definition of 'quantity value', i.e., an equality-by-definition.)

The relation between a-quantities and c-quantities can be used now to clarify both the structure of the process of measurement and the way in which such process provides information on the object under measurement in terms of a γ-equality, as shown in Table 1.

Table 1 – Synopsis of the relations between quantities and quantity values.

Addressed quantities,	Classifier quantities,
such as the length of this object,	such as 0.123 m,
are elements of the world,	are elements of a classification,
are assumed to be unknown before measurement,	are assumed to be known before measurement,

are individuated in terms of a given object under measurement	are individuated independently of any object under measurement
as measurands	as quantity values
that are represented by quantity values.	that represent measurands.

This can be further synthesized in the following:

Thesis 2: *there is a complementarity in measurement between a-quantities and c-quantities*.

3.2. Measurement as assignment of quantity values to quantities

This conceptual framework gives a basic justification of the epistemic role of measurement, thus interpreted as a process in which an a-quantity, which is known by address, i.e., by reference to the object under measurement in as much as it is empirically given, turns out to be known also by value, i.e., by reference to a class in a given classification. Accordingly, measurement is abstractly modeled as involving four stages (for the sake of simplicity here presented only in the case the additive concatenation $\oplus$ is applicable):

- Stage 1 – *definition of the measurand*:

$q = Q(\omega)$ // equality is equality-by-definition here

the quantity q is specified as the a-quantity $Q(\omega)$ of a given object ω;

- Stage 2 – *measurement standard calibration*:

$Q(\underline{\omega}) =_\gamma 1 \cdot [q]$ // γ-equality

a given object $\underline{\omega}$ is recognized as a standard realizing the unit $[q]$, i.e., the quantity $Q(\underline{\omega})$ = $1 \cdot [q]$, and then $Q(\underline{\omega}) \oplus Q(\underline{\omega}) = 2 \cdot [q]$, and so on;

- Stage 3 – *experimental comparison*:

$Q(\omega) \approx Q(\underline{\omega}) \oplus Q(\underline{\omega}) \oplus ...$ repeated $\{q\}$ times   // α-equality

$Q(\omega)$ is recognized as indistinguishable from the a-quantity corresponding to a set of $\{q\}$ replicas of the object $\underline{\omega}$;

- Stage 4 – *representation of the measurand*:

$Q(\omega) =_\gamma \{q\} \cdot [q]$ // γ-equality

the measurand, i.e., the a-quantity $Q(\omega)$, is finally represented by a quantity value, i.e., the c-quantity $\{q\} \cdot [q]$.

This description clearly shows that it is Stage 2, measurement standard calibration, where the relation between an a-quantity, $Q(\underline{\omega})$, and a c-quantity, $1 \cdot [q]$, is first identified. This critical relation is entirely based on the fact that calibration is performed under controlled conditions on the classification generator $[q]$ and the derived quantities $k \cdot [q]$. Furthermore, Stage 2 assumes that a unit $[q]$ for Q was previously defined, as the individual quantity of an object $\underline{\omega}$:

$Q(\underline{\omega}) = 1 \cdot [q]$

e.g., length(path traveled by light in vacuum during a time interval of 1/299 792 458 of a second) = 1 m, where the fundamental fact here is that this is a *defined* (instead of obtained) γ-equality, thus highlighting that the experimental identification of an a-quantity (the measurand) with a c-quantity (the measured quantity value) obtained by measurement is grounded on a definitional identification of an a-quantity (the quantity unit, as realized by a measurement standard) with a c-quantity (the unit quantity value), typically through a traceability chain.

In the empirical component of measurement, Stage 3 (of course, the experimental comparison is seldom performed synchronously, as in the case of the two-pan balance, and more often it is obtained indirectly via a calibrated transducer), the information acquired in calibration is complemented with information on the measurand, so that the inferential assumption is finally made, Stage 4, that such information can be represented as in eq. (1). Hence, through definition and calibration the inter-subjective role of measurement is justified, aimed at providing information that can be communicated by means of c-quantities, i.e., quantity values, whereas through comparison the empirical role of measurement is ensured, aimed at providing information that can be used to get knowledge on some aspects of the world.

3.3. A generalization: a-properties and c-properties

The proposed interpretation of the relation between quantities and quantity values is in fact independent of the algebraic structure, i.e., the "scale type", of the involved entities: what has been considered in terms of quantities and quantity values can be applied not only to ordinal quantities, where the concept of quantity unit is not defined [1, 1.26], but also to generic properties and property values, thus significantly widening the scope of this conceptual framework. Indeed, eq. (1) is straightforwardly generalized to:

$$p = \{p\} \text{ in } [p] \tag{3}$$

where:

- p is an addressed individual property, e.g., the color of the surface of a given object;

- $[p]$ is a classification related to the general property P, e.g., [red, orange, yellow, green, blue, violet];

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- {p} is a class in [p], e.g., blue,

so that the equation is read, e.g., the color of the surface of this object is blue in the classification [red, orange, …] (when the classification is presented as a scale, the property value is sometimes reported as "{p} on [p]", as in the case "the hardness of graphite is 1,5 on the Mohs' scale").

The previous analysis about individual quantities in eq. (1) can now be generalized to individual properties in eq. (3). In particular, the operational interpretation of eq. (3) is that at least in principle there exists a procedure allowing to generate an object that is an element of the class {p} in the classification [p], and that such object is indistinguishable from the object ω by a *P*-related comparison, so that the individual property $p = P(\omega)$ is precisely {p} in [p]. This shows that numbers do not play any essential role in this conceptualization, and that the concept 'property value' can be consistently characterized, e.g., as the element of a given classification representing a property. Whenever the property evaluation is of algebraically weak type (typically nominal or ordinal) [27], so that no property related concatenation is allowed, the classification can only be extensionally defined, i.e., by explicitly listing the individual properties of which it is constituted. In the ratio case, whose algebraic structure includes concatenation, the classification can be generated by a quantity unit, thus re-obtaining the definition of '(ratio) quantity value', "unit multiplied by a number representing a quantity".

4. Magnitudes

The upshot of the previous section is fourfold:

1. both measurands and quantity values are quantities;

2. measurands are a-quantities and quantity values are c-quantities;

3. measurement consists in identifying a c-quantity with an a-quantity;

4. such identification is based on the definitional identification of an a-quantity with a c-quantity, as obtained by a quantity unit definition.

This provides a consistent account of measurement as a process of assigning a quantity value to a given measurand (as mentioned in the Introduction, measurement uncertainty is omitted in this analysis: more generally, the assignment is of "one or more quantity values" [1, 2.1]). However, a further question can be posed concerning the interpretation of the three relations of α-equality, β-equality, and γ-equality, whether they are to be conceived in terms of *identity* or not. In particular, given that, e.g., the length $Q(\omega_i)$ of object ω_i is ascertained to be equal to the length $Q(\omega_j)$ of object ω_j, the question is whether such α-equality implies that $Q(\omega_i)$ and $Q(\omega_j)$ are one and the same entity, or they are two distinct, although equivalent, entities. In the philosophical lexicon, the alternative is about a-quantities as *either universal or particular* entities. This issue is indeed clearly a philosophical one, with negligible operative consequences, and nevertheless it deserves some consideration here because such two scenarios have significant consequences on the interpretation of the concept of magnitude, which is sometimes used to describe measurement and measurement results.

In what follows we will make use of the (philosophical) distinction between the *reference* of an expression and its *sense*, conceived as the way in which the reference is presented by that expression (a classical analysis on the subject is in [28]; in view of the present discussion the first sentence of this paper is particularly interesting: "Equality gives rise to challenging questions which are not altogether easy to answer."). For example, the expressions "6+5" and "$121^{1/2}$" have the same reference, the number 11, yet they have

different senses since they present it as the sum of 6 and 5 and the square root of 121 respectively. Indeed, one could grasp the first expression only and nevertheless the reference remains the same. This distinction enables us to account for situations where different quantity values are equated or are attributed to the same measurand as being different ways of presentation of the same quantity.

On this basis let us take into account the problem: should the relations of α-equality, β-equality, and γ-equality be conceived as identities or not?

First of all, both hypotheses share the same position as for β-equality: a sentence such as "$2 \cdot 10^{-2}$ m $=_\beta$ 0.7874... in" is interpreted as stating that both "$2 \cdot 10^{-2}$ m" and "0.7874... in" refer to the same quantity q, even if they have different senses, since they refer to q by representing it as the quantity related to the classes $2 \cdot 10^{-2}$ and 0.7874... in the classifications of meters and inches respectively. Hence, *one* quantity q is involved in the relation with *two* different classifications.

On the other hand, α-equality and γ-equality can be differently interpreted.

Scenario 1: all quantities, both a-quantities and c-quantities, are universal entities.

(i) The sentence "$Q(\omega_i) =_\alpha Q(\omega_j)$" is interpreted as stating that the references of "$Q(\omega_i)$" and "$Q(\omega_j)$" are the same universal quantity q, even if "$Q(\omega_i)$" and "$Q(\omega_j)$" have different senses, since they refer to the quantity q by representing it as the one exemplified by the objects ω_i and ω_j respectively. Hence, *one* quantity q is involved in the relation with *two* different addresses, ω_i and ω_j.

(ii) Accordingly, the sentence "$Q(\omega) =_\gamma 2 \cdot 10^{-2}$ m" is interpreted as stating that the references of "$Q(\omega)$" and "$2 \cdot 10^{-2}$ m" are the same universal quantity q, even if "$Q(\omega)$"

and "$2 \cdot 10^{-2}$ m" have different senses, since "$Q(\omega)$" refers to the quantity q by representing it as the one exemplified by ω, while "$2 \cdot 10^{-2}$ m" refers to quantity q by representing it as the one related to the element $2 \cdot 10^{-2}$ in the classification of meters. Hence, *one* quantity q is involved in the relation – the measurand is identical with the quantity value – with *two* different epistemic states – before measurement the measurand is known only by address whereas the quantity value is known by value.

In this Scenario, synthesized in Fig. 2, empirical indistinguishability is interpreted as an effect of the identity between quantities and there is no need to introduce a further concept 'magnitude', as it would be identically *magnitude*$(Q(\omega)) = Q(\omega) = q = 1$ m.
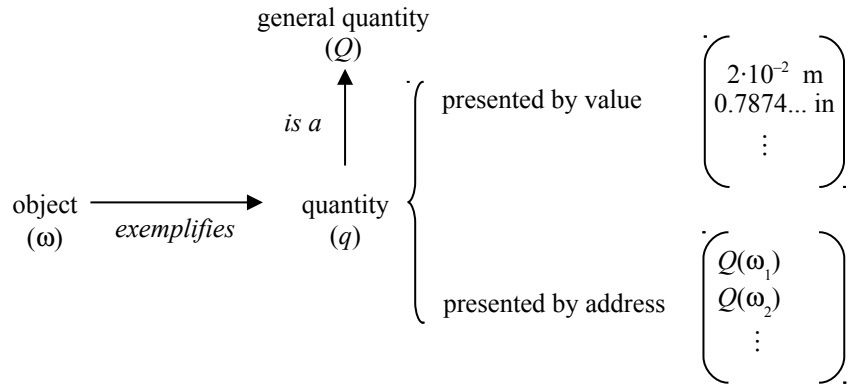

Figure 2 – Relations between a-quantities and c-quantities according to Scenario 1.

Scenario 2: there are universal and particular quantities; a-quantity are particulars and c-quantities are universals.

(i) The sentence "$Q(\omega_i) =_\alpha Q(\omega_j)$" is interpreted as stating that the references of "$Q(\omega_i)$" and "$Q(\omega_j)$" instantiate the same universal quantity q, even if they have actually different references, a particular quantity q_i possessed by ω_i and a particular quantity q_j possessed by ω_j respectively. Hence, *two* quantities are involved in the relation.

(ii) Accordingly, the sentence "$Q(\omega_t) =_\gamma 2{\cdot}10^{-2}$ m" is interpreted as stating that the reference of "$Q(\omega_i)$", i.e., the particular quantity q_i, is an instance of the reference of "$2{\cdot}10^{-2}$ m", i.e., the universal quantity q. Hence, *two* quantities are involved in the relation: the measurand is not identical with the quantity value, being a particular and not a universal entity.

In this Scenario, synthesized in Fig. 3, empirical indistinguishability is interpreted as an effect of the fact that particular quantities are different, and there is room to introduce a concept 'magnitude' as the universal quantity q: indeed, $Q(\omega_i) \approx Q(\omega_j)$ if and only if *magnitude*($Q(\omega_i)$) = *magnitude*($Q(\omega_j)$), where "$\approx$" denotes such indistinguishability relation.
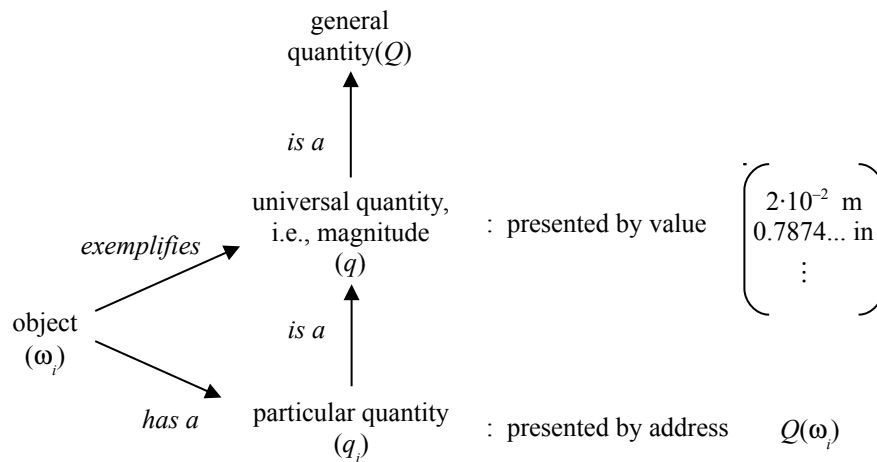

Figure 3 – Relations between a-quantities and c-quantities according to Scenario 2.

(It is interesting how such two Scenarios are presented by Russell in the foundational text [29], where they are called the "relative" and the "absolute" theory of magnitudes respectively: "The relative theory regards equal quantities as not possessed of any common property over and above that of unequal quantities, but as distinguished merely by the mutual relation of equality. There is no such thing as a magnitude, shared by equal

quantities." "In the absolute theory, there is, belonging to a set of equal quantities, one definite concept, namely a certain magnitude. [...] Two magnitudes cannot be equal, for equality belongs to quantities, and is defined as possession of the *same* magnitude. [...] The quantities which are instances of a magnitude are particularized by spatio-temporal position". The classical philosophy of measurement did not reach a common understanding on this alternative. For example while Russell supported the absolute theory, on this matter Nagel wrote "if Occam's razor still can cut, the magnitudes demanded by the absolute theory may be eliminated" [30]. And more recently on the same vein Kyburg asserted: "We already have an uncountable number of real numbers in our universe; why multiply these entities by supposing there is also an uncountable number of magnitudes of length, of distance, of temperature, etc.?" [31].)

The definition of 'quantity value' proposed in the VIM3, English version ("number and reference together expressing magnitude of a quantity"), is consistent with Scenario 2, provided that "expressing" is interpreted as "representing". Indeed:

(1) number and reference together representing the magnitude of a quantity

coincides with:

(1*) representation of the magnitude of a quantity as number and reference

i.e., according to Scenario 2:

(1**) representation of the magnitude q of a quantity q_i as $\{q\}$ and $[q]$

where q is then a universal quantity and q_i is a particular quantity.

It is interesting that the French version of the definition is neutral with respect to Scenarios 1 and 2, provided that, as above, "expressing" is interpreted as "representing":

(2) number and reference together constituting the quantitative representation of a quantity / magnitude

coincides with:

(2*) representation of a universal quantity / magnitude by means of number and reference

i.e.:

(2**) representation of the universal quantity / magnitude q by means of $\{q\}$ and $[q]$

where no references are made to any particular quantity.

5. Conclusions

The theses presented in this paper propose a consistent conceptualization of 'quantity' and 'quantity value', and actually and more generally of 'property' and 'property value'. They provide:

i) a justification of the hypothesis that quantities and quantity values are entities of the same sort, i.e., individual quantities;

ii) a distinction between individual quantities known by address (a-quantities), such as measurands, and individual quantities known by value (c-quantities), such as quantity values, thus grounding a sound interpretation of the basic formula of quantity calculus, $q = \{q\} \cdot [q]$ and giving an unequivocal characterization of the concept 'quantity value' as c-quantity, where the quantity unit is the generator of a classification;

iii) a generalization of these concepts to properties and property values;

iv) an insight into the conceptual structure of measurement as a process aimed at exploiting the complementarity between a-quantities and c-quantities, by merging the

information given by measurement standard calibration with the information obtained experimentally on the measurand;

v) finally, a distinction between two possible scenarios according to which the concept of magnitude can be interpreted.

On this ground, the sketch of definitions for the relevant concepts can be tentatively proposed as follows (only for quantity-related concepts: the generalization to properties is straightforward; these definitions hold identically for both Scenario 1 and Scenario 2 except where noted).

general property: <primitive concept>

general quantity: *general property which, according to the current state of knowledge, can be evaluated at least in an ordinal scale* [VIM3-compliant version] [or: *at least in an interval scale*; traditional version]

individual quantity: *instance of a general quantity*

addressed quantity (a-quantity): *individual quantity specified by an object which instantiates it* [or: *which possesses it*; Scenario 2]

classifier quantity (c-quantity): *individual quantity specified as element of a classification*

measurand: *a-quantity intended to be measured*

quantity value: *c-quantity aimed at representing an a-quantity* [or: *representing the magnitude of a-quantity*; Scenario 2]

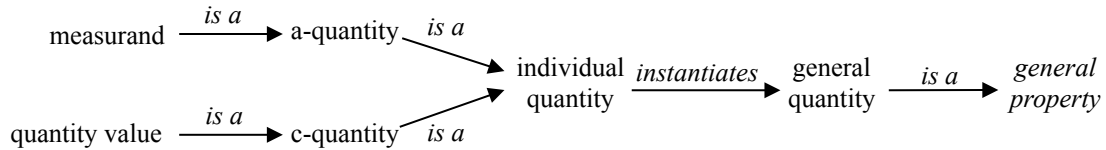According to these definitions such concepts present a remarkable symmetry, as shown in Fig. 4 (for Scenario 1):

Figure 4 – Relations between the basic concepts according to the proposed definitions.

At least a basic issue still remains undiscussed in this conceptual framework, whether any individual quantity is either an a-quantity or a c-quantity, or there may be individual quantities which are neither a-quantities nor c-quantities. This apparently abstract question becomes relevant when the nature of quantity units, a specific case of individual quantities, is investigated. They are a-quantities when defined in reference to a measurement standard which realizes them. It is the case of the unit of mass, defined as "equal to the mass of the international prototype of the kilogram" [24], where such prototype is the address of the a-quantity kilogram. But what about a quantity unit defined in reference to a phenomenon which is assumed to be a universal entity because of a physical law? Consider the case of light in vacuum for the meter: of course, many instances of light in vacuum can be obtained, and exploited as measurement standards, but the meter is defined in reference to the phenomenon as such, not to any of such instances. And what about quantity units possibly defined in reference to universal constants? Do these definitional changes also imply ontological changes? This opens the door to a further discussion on the subject.

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