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Property evaluation types

A. Giordani[1], L. Mari[2*]

[1] Università Cattolica, Largo Gemelli, 1 – 20100 Milano, Italy; alessandro.giordani@unicatt.it

[2] Università Cattaneo – LIUC, C.so Matteotti, 22 – 21053 Castellanza (Va), Italy; lmari@liuc.it

Abstract

An appropriate characterization of property types is an important topic for measurement science. On the basis of a set-theoretic model of evaluation and measurement processes, the paper introduces the operative concept of *property evaluation type*, and discusses how property types are related to, and in fact can be derived from, property evaluation types, by finally analyzing the consequences of these distinctions for the concepts of 'property' used in the *International Vocabulary of Metrology – Basic and General Concepts and Associated Terms* (VIM3).

Keywords: Measurement science; Property type; Measurability.

1. Introduction

"In communication, not every individual object in the world is differentiated and named. Instead, through observation and a process of abstraction called conceptualization, objects are categorized into classes, which correspond to units of knowledge called *concepts*, which are represented in various forms of communication." [1]. According to a standard view in philosophy of science, originated by the seminal study by Campbell [2] and developed within the positivist tradition, in particular [3-5], the concepts used by scientists to describe objects are classified, according to three general types, as qualitative, comparative, and quantitative, as sketched in Figure 1.

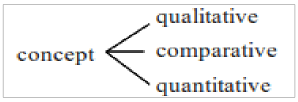

Figure 1. A traditional classification of concepts.

Qualitative concepts simply allow to categorize objects in classes, so that a system of such concepts generates a classification of a domain of objects. Comparative concepts allow to relate objects with each other in virtue of a relation of total order. Finally, quantitative concepts allow to assign numerical values to objects in such a way that the relations between numbers represent relations between objects.

In his 1873 *Treatise on Electricity and Magnetism,* J. C. Maxwell [6] characterized a specific kind of entities, falling under quantitative concepts, that he called "physical quantities", as entities that can be represented by two components: (i) a measurement unit; (ii) a numerical value assigned with

* Corresponding author.

respect to the unit[1]. Building on the Euclidean concept of measure (see, e.g., [8]), the measurement unit was assumed to be a known quantity of the same kind of the quantity to be expressed, the numerical value describing the number of times the unit is to be taken in order to make up the quantity. This can be synthesized in the form:

$$Q = \{Q\}[Q]$$

where Q is the quantity to be expressed, $\{Q\}$ is a number and $[Q]$ is a measurement unit. This is now a standard expression within the so called *quantity calculus*. If written:

$$\{Q\} = Q/[Q]$$

this expression justifies why entities such as Q are called "ratio quantities" (for a thorough analysis on this subject see [9]).

The characterization provided by Maxwell gives rise to a number of questions when confronted with the aforementioned classification of concepts:

1) What kind of entities are the quantities?

2) What kind of entities are the kinds of quantities?

3) How is it possible to attribute the same kind to different quantities?

4) Is it possible, and in the case how, to generalize the expression $Q = \{Q\}[Q]$ to all kinds of concepts, qualitative and comparative as well as quantitative?

The significance of last question is highlighted by the comparison of examples such as:

- the length of this object is 1.23 m;
- the shape of this object is E3 in a given classification system,

that, despite of the difference of the involved concepts, have an invariant structure:

property of *object* is *value* with respect to *reference*.

A significant step towards both the generalization of Maxwell's standpoint and the solutions of the foregoing problems was made by the *International Vocabulary of Metrology – Basic and General Concepts and Associated Terms* (VIM3) [10]. The VIM3 proposes a threefold division for the primitive superordinate[2] concept of property[3], similar to the traditional one. This concept is firstly specified into the two subordinates 'nominal property' and 'quantity', the differentiating

[1] When dealing with concept systems it is appropriate to maintain the distinction between *representation* and *expression*: the fact that an entity can be represented by a number and a measurement unit is independent of the expression chosen for the number, i.e., a numeral, and the term by which the unit is identified. On the basis of this understanding, one can then write, e.g., <the quantity *Q*> as a shortcut for <the quantity identified by the term "*Q*">. For a thorough consideration of this topic in the field of measurement, see [7].

[2] See [1, 5.5.2.1]: "In a hierarchical relation, concepts are organized into levels of superordinate and subordinate concepts. For there to be a hierarchy, there must be at least one subordinate concept below a superordinate concept.". In the VIM3 'property' is among the undefined, and therefore "primitive", concepts.

[3] The choice of assuming 'property' as a primitive concept in a document as the VIM3 is a wise one, since *what a property is* is a fundamental ontological problem still object of lively discussions among philosophers ("Questions about the nature and existence of properties are nearly as old as philosophy itself." [11], that includes a good bibliography on the subject). Furthermore, the concept of property seems to be not definable in terms of more primitive concepts, given the difficulty of formulating a definition for 'property' that does not include occurrences of terms such as "characteristic", "attribute", "feature", and the like (see, e.g., the definition proposed in [12]: "inherent state- or process-descriptive feature of a system including any pertinent components"). It can be noted that the standard [13], that includes definitions for concepts such as 'object' and 'concept', uses 'property' *but does not define it*.

characteristic being that the latter "has a magnitude"[4]. In its turn, the concept of quantity is specified into 'ordinal quantity' and 'quantity with unit',[5] that finally includes 'quantity of dimension one'[6] as a specific case. These relations are presented in Figure 2.

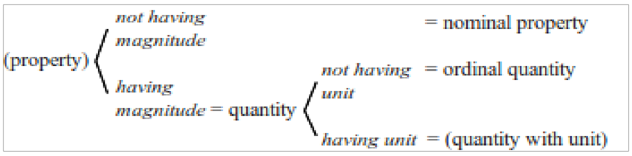

Figure 2. 'property' and some of its subordinate concepts according to the VIM3.

This structure implies that such concepts are pairwise mutually exclusive: a nominal property is not a quantity and a quantity is not a nominal property. Furthermore, because of the negative connotation of 'nominal property' as having no magnitude, they are exhaustive: by definition, each property is either a nominal property or a quantity. The same conclusion can be drawn for 'ordinal quantity' and the undefined 'quantity with unit' (Dybkaer [12] uses "unitary quantity"): an ordinal quantity is not a quantity with unit and a quantity with unit is not an ordinal quantity, and each quantity is either ordinal or with unit.

The above mentioned philosophical perspective and the position underlying the VIM3 definitions are clearly related: qualitative concepts correspond to nominal properties, while comparative and quantitative concepts correspond to ordinal quantities and quantities with units respectively. In this sense, the VIM3 builds on the tradition and, at the same time, offers a noteworthy contribution in the development of the Maxwell's standpoint, by embedding that philosophical position into the metrological framework, traditionally focused on quantities with units, as witnessed by the critical role played by measurement units in the SI [14].

The basic motivation of the present paper is the belief that the VIM3 has taken the right direction, but also that in this evolutionary path some further steps may be done. Under the assumption that a good foundation of measurement science requires a sound categorization of properties, the present contribution to this path is grounded on two general claims:

- first, the classification of properties as nominal, ordinal, and so on, should be based on the operative features of the processes by which properties are evaluated, and only in a derived and

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[4] As stated by the VIM3, definition 1.1, a quantity is a property of an entity, "where the property has a magnitude that can be expressed as a number and a reference". According to definition 1.30, a nominal property is a property of an entity "where the property has no magnitude".

[5] As stated by the VIM3, definition 1.26, an ordinal quantity is a "quantity, defined by a conventional measurement procedure, for which a total ordering relation can be established, according to magnitude, with other quantities of the same kind, but for which no algebraic operations among those quantities exist". Furthermore, ordinal quantities "have neither measurement units nor quantity dimensions" [1.26 Note 1]: hence, having unit can be assumed to be the characteristic distinguishing quantities other than ordinal from ordinal quantities. The VIM3 does not assign a term to quantities with unit: only in the concept diagram A.1 of Annex A the term "quantity expressed by a measurement unit" appears, thus acknowledging that it refers, peculiarly, to a primitive concept.

[6] As stated by the VIM3, definition 1.8, a quantity of dimension one is a "quantity for which all the exponents of the factors corresponding to the base quantities in its quantity dimension are zero". This refers to definition 1.7, according to which a quantity dimension is an "expression of the dependence of a quantity on the base quantities of a system of quantities as a product of powers of factors corresponding to the base quantities, omitting any numerical factor".

specific sense considered a feature of properties “in themselves”;

- second, despite of its being grounded in tradition, a structure *by oppositions* to define the relevant subordinate concepts of ‘property’ should be complemented with a structure *by inclusions*, and actually derived from it.

This paper is devoted to justify these claims, in the context of a specific mathematical framework. In synthesis, we are going to propose a primitive concept of *property evaluation type*, PET for short, such that a concept of property type can be derived by the one of PET under specified conditions (our first general claim), and different PETs are related with each other by the supertype-subtype relation (our second general claim). As a result, a uniform solution to the four problems presented above is obtained. The basic idea is that:

1) quantities, and specifically quantities with unit, are individual properties that can be determined by means of procedures characterized by a given PET;

2) kinds of quantities, and specifically unitary kind of quantities, are general properties attributed to objects on the basis of the available comparison procedures characterized by a given PET;

3) the same kind is attributed to different properties when they can be distinct with each other by means of one of such procedures;

4) the expression $Q = \{Q\}[Q]$ is generalizable as:

$$P = \{P\}[P]$$

where P is the property to be evaluated and $\{P\}$ is a conceptual way to identify a class within a reference classification $[P]$, thus under the assumption that in the general case the term $\{P\}[P]$ has to be read “$\{P\}$ in reference to $[P]$”, not “$\{P\}$ times $[P]$”.

Accordingly, the choice of restricting the scope of measurement to specific types is mainly conventional: the evolutionary path driven by the VIM3, that already departed from the Euclidean tradition by including ordinal quantities among the measurable entities, may consistently lead to acknowledge that property measurability is not affected by property types.

2. Preliminary analysis

2.1. The general view

The VIM3 is undoubtedly a reference document, if not *the* reference document, to set a metrologically-oriented concept system on properties, and properties have been the subject of some of the most significant changes with respect to the second edition of the VIM (VIM2, [15]). The modifications introduced to the Maxwell’s standpoint, exclusively devoted to ratio quantities as the sole important properties in the context of physical sciences, become apparent, for example, in the definition of ‘quantity value’. Where the VIM2, definition 1.18, has “magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number”, the VIM3, definition 1.19, states “number and reference together expressing magnitude of a quantity”. The important point here is the greater generality of the new definition, obtained by replacing “a unit of measurement multiplied by a number” with the generic a “number and reference”, a condition for including ordinal quantities, a concept outside the VIM2, among the measurable properties.

The issue is whether the conceptual framework on measurability obtained by this extension is stable in its bases and consistent in its scope and structure: since measurability is historically associated to *quantitative* concepts only, how its extension to *comparative* concepts is justified? And why *qualitative* concepts are instead still outside the scope of measurement?

Several reasons highlight the importance of these questions, and among them:

- from a conceptual point of view, the VIM3, 2007-2008, has just changed a tradition[7] dating back many centuries (consider that the VIM2, 1993, still did not include ordinal quantities in its scope): has this historical transition been completed?
- from an operative point of view, in several fields the requirement is emerging to adopt principles and procedures for properly assigning values, including uncertainty estimations, to nominal properties: should this effort be performed inside the metrological context?

2.2. The specific view

The VIM3 assumes a clear-cut distinction among the direct subordinate concepts of ‘property’ relating to measurement, as “measurement does not apply to nominal properties” [2.1 Note 1]. The justification of this position has at least two points worth of some analysis.

First, while the necessary condition required by the VIM3 for measurability is ‘having a magnitude’, nothing is said about what a magnitude is. That “a magnitude [...] can be expressed as a number and a reference” [1.1] (or, better, “expressed as a numeral and the name of a reference”, as commented above) is not so helpful on this matter, as it is a stipulation (i) related to how a magnitude can be *expressed*, not what it *is*, and (ii) stated in conditional form (“*can* be”), not as a requirement (“*must* be”). If one does not understand what a magnitude is, then the definition is not applicable, since every property can be, in principle, expressed by means of a numeral and the name of a reference[8], and, if one does understand the concept of magnitude, then that part of the definition is not relevant, since adding that it can be expressed in a particular way is simply useless for determining the concept. The point here is not that some definitions refer to undefined concepts, an unavoidable situation as far as circular definitions are not allowed and the substitution principle

[7] Or has it acknowledged and embodied such change? The issue whether the VIM should drive or follow the evolution is a complex one, also for its political implications on the role of the *Joint Committee on Guides in Metrology* (JCGM), and more generally of standardization bodies, in a rapidly changing (scientific, technological, economical, social) world.

[8] To assert that a property can be expressed by means of a numeral and the name of a reference does not imply that the numeral in the expression actually denotes a number. In fact, the justification of the classification of kinds of properties introduced below is strictly connected to the identification of criteria for determining how numerals used to represent properties are to be conceived.

(VIM3, Conventions – Terminology rules) is maintained. Rather, the effort should be done to maximize the chance of the widespread understanding of such an important subject. The issue whether a better characterization can be adopted then arises.[9]

Second, the criteria characterizing property types, i.e., their delimiting characteristics [13], remain ambiguous if inferred from the definitions of the specific types as they are given in the VIM3: does the type of a property depend on the way the property is measured or examined (as it is stated for ordinal quantities, that are "defined by a conventional measurement procedure"), or is it instead a characteristic of the property "in itself" (as in the case of nominal properties, that "have no magnitude")? It seems that definitions should be as consistent as possible.

For the sake of generality and to avoid premature conclusions, in the following we will discuss not only of measurement of quantities but, in a more generic sense, of *evaluation of properties*, where the concept of evaluation is assumed with the customary meaning of 'assignment of values', and therefore in this case 'assignment of property values'.[10] Such an evaluation will be interpreted here in functional terms, i.e., as a process:

- that requires an entity to be *examined* to acquire information about a given property,
- which is then *determined*,
- and finally *represented by a value* that has to be *expressed* to be communicated.

These three steps – examination, determination, representation – seem to be properly designable as "evaluation", a term accounting for the whole process and involving the concept of determination resulting from examination.

Our strategy will be of characterizing a concept of *property evaluation type* (PET), as derived from the so-called "theory of scales of measurement", originally introduced by S. S. Stevens [16] and rooted in the works by H. Helmholtz [17] and O. Hölder [18]. This theory is adopted here according to an interpretation that is quite far from the original presentation made by Stevens not only because of the avoidance of the term "scale", whose applicability in the case of nominal properties is questionable as such concept seems to imply an ordered sequence of entities, but also, and more relevantly, because of the mentioned crucial choice to relate the theory itself not specifically to measurement, but more generically to evaluation.[11]

[9] The hypothesis that property types can be defined with no reference to magnitudes is supported by the fact that in the French definitions of the VIM3 the concept 'property having a magnitude expressed as a number and a reference' (definition 1.1) – a triadic relation <property, magnitude, number_and_reference> – is rendered as 'property quantitatively expressed as a number and a reference' – a diadic relation <property, number_and_reference> –, also considering that the adverb "quantitatively" does not seem adding any specific information in this case.

[10] The VIM3 uses "examination" (without defining it) to designate the assignment of nominal property values. Its characterization of property types by opposition plausibly implies that a measurement *is not* (a specific case of) an examination, as a nominal property is not a quantity (on this matter the only reference in the VIM3 is a note to the definition of 'reference material' [5.13 Note 8]: "ISO/REMCO has an analogous definition [45] but uses the term "measurement process" to mean 'examination' (ISO 15189:2007, 3.4), which covers both measurement of a quantity and examination of a nominal property.": not a so clear standpoint, also because referred to a different standard...). This further justifies our proposal of adopting "evaluation".

[11] While the Stevens' theory has received several substantial criticisms, most of them are related to his claim that the theory concerns measurement, of which he had a very general concept ("measurement is, in the broadest sense, defined as the assignment of numerals

The choice of focusing on property *evaluations* instead of properties derives from the general consideration that what characterizes a property as object of a given evaluation does not depend only on the property "in itself", but also, critically, on the evaluation. For example, length, a paradigmatic case of a quantity with unit, can be measured by means of a system able to acquire only ordinal information[12]: from the point of view of such an evaluation, length behaves as an ordinal quantity and as such its values should be dealt with in this particular case. Hence, a concept of property type can be defined, as we will see in the following, but only in reference to the types of its available evaluations.

The concept of PET will be formulated here in a set-theoretical framework, under the simplifying assumption that uncertainties can be neglected. Given the starting point of this paper – characterizing the mutual relations among the subordinate concepts of 'property' – this seems to be acceptable (the VIM3 assumes the same position, as the definitions of 'quantity', 'ordinal quantity', and so on, are unrelated to (measurement) uncertainty).

The step-by-step discussion that follows is aimed at making the standpoint understandable in its conceptual bases and at deriving from it the criteria required to properly characterize the relevant PETs, so that a possible set of related definitions may be consistently formulated from the outcome of the analysis. The next Section will introduce the preliminary concepts required to deal with PETs, under the simplifying assumption that evaluations are purely classificatory, i.e., only nominal properties are taken into account. The concept of PET will be built on this basis in Section 4 and discussed in its consequences in Section 5. Because of this presentation strategy, Section 3 will deal with sets and functions only, that will be respectively generalized to relational systems and morphisms in the subsequent Sections.

3. Towards a concept of property evaluation type (PET)

3.1. The conceptual framework

In agreement with the standard [1], the starting point of the analysis is the assumption that objects are characterized by individual properties and classified under kinds of objects, i.e., concepts construed as units of knowledge. In their turn, concepts are characterized by kinds of properties, under which individual properties are subsumed.[13] Thus, a given mechanical object is characterized

to objects and events according to rule" [16]). As re-interpreted in this paper, the theory avoids these critical issues.

[12] An example of "purely ordinal length" can be presented as follows. First of all, a set of sticks is prepared such that (i) if aligned on one tip no two sticks have the opposite tip aligned (a nominal information), and (ii) the sticks are ordered so that each stick exceeds the previous one if compared by means of the same procedure (an ordinal information). Then the ordinal numbers 1, 2, .., are assigned to the sticks as appearing in the obtained sequence. On this basis, the measurement is performed by comparing the object whose length has to be measured with each stick in turn according to the same procedure, and identifying the pair of sticks such that the object exceeds the first stick and does not exceed the second stick (an analogous procedure would allow calibrating the set of sticks with respect to a given standard set). The measured quantity value could be, conventionally, either of the ordinal numbers assigned to such sticks, together with an identifier of the sequence of sticks and the given comparison procedure. The reader will have noted the similarity, if not the isomorphism, of this procedure with the measurement of hardness in Mohs' scale.

[13] Some further analyses on the complex concept of kind of property / kind of quantity can be found in [12], [19] and [20].

by its individual position and momentum and could be classified as a point mass, where the concept of point mass is characterized by position and momentum. The diagram in Figure 3 depicts these concepts and their relations:

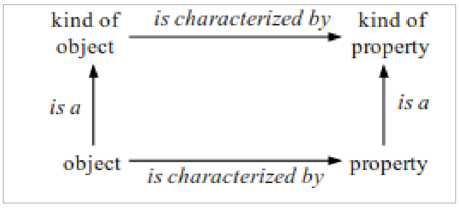

Figure 3. Concepts and their relations in the adopted conceptual framework.

Accordingly, once classified under a given kind, an object can be evaluated with respect to any kind of property characterizing its kind, i.e., property values can be assigned to represent it. Therefore, an evaluation process can be abstractly conceived as a black box taking as input an object, that can be called the object under evaluation, or more specifically the object under measurement,[14] and producing a value as output. So conceived, the process can be modeled by a mapping:

f:A→V

from a domain A of objects to a range V of values. In this sense, the basic entities grounding the concept of PET are such mappings f, intended as mathematical models of evaluation processes. This modeling only assumes that:

- A is a domain of empirical objects to be evaluated with respect to a kind of property;
- V is a set of values representing the individual properties of the given kind;
- f is a function that maps objects under evaluation to values, in such a way that the assigned values provide information on the empirical relations between the objects[15] [21].

The domain of f is to be intended here as the set of objects that differently instantiate a given kind of property, i.e., as *a set of objects exhibiting differences under a given respect*:

- the respect is the relevant kind of property, i.e., a general (or determinable) property;
- the differences are grounded on the properties of the objects, i.e., individual (or determinate) properties.

Hence, that a kind of property is differently instanceable is a necessary condition for evaluation.[16]

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[14] Hence an object is here what the VIM3 calls a "phenomenon, body, or substance" [1.1]. We are using this term in a general sense as referring to objects and processes, both physical and social ones, that are accessible from an intersubjective point of view.

[15] Note that the last step of expressing values in a suitable symbolic form is not taken into account here, for its purely linguistic nature and its lack of modeling implications.

[16] The distinction between quantities in general sense, i.e., general quantities or kinds of quantities, and quantities in particular sense, i.e., particular or individual quantities, is well known in metrology (see, e.g., [22] p. 30 and [23] pp. 65-67). A generalization of this distinction to properties is proposed here.

A philosophical issue arises here, as the distinction between *individual* and *general* properties should not be confused with the distinction between *particular* and *universal* properties. Hence: is the property of an object to be interpreted as an universal or a particular entity? For example, the assertion that two objects can have the same length, or the same shape, can be interpreted in either of two ways: (i) there is just one length, a universal one, that characterizes both objects; (ii) there is a length, a particular one, that characterizes one object, and such length is perfectly indistinguishable from the length of the other object. In compliance with a common conceptualization and terminology in measurement science (see, once again, for example [22] and [23]), we opt to consider both individual and general properties as universals, possibly characterizing many objects. In this sense, an individual property is just the

Note that the involved kind of property could be, in principle, the very property of interacting with the instruments used to perform the evaluation process. Thus, the fact that f is the model of an empirical evaluation applied to a domain determined by a certain kind of property is consistent with the option that A is determined in the definition of the function f itself. In this sense, the domain A could simply be defined as the set of the objects to which the evaluation process can be empirically applied.[17]

3.2. The mathematical framework

A few elementary results of function theory can be helpfully recalled to better highlight how an evaluation process can be modeled by means of a function.

- Any function f:A→V induces an equivalence relation, and thus a partition, i.e., a set of equivalence classes, on its domain A, such that any two elements a_i, $a_j \in$A belong to the same equivalence class, $a_i \approx_f a_j$, if and only if they are indistinguishable with respect to f, i.e., they are evaluated in the same way, $f(a_i) = f(a_j)$. As a consequence, the equivalence relation $\approx_f$ can be thought of as meaning, trivially, "having the same value according to f".[18] The set of equivalence classes is called the quotient set of A under f and is customarily denoted as A/$\approx_f$.
- Any function f:A→V can be viewed as the composition of two functions: a surjective function f_α:A→A/$\approx_f$, mapping elements of A onto equivalence classes in A/$\approx_f$, and an injective function f_β:A/$\approx_f$→V, mapping such classes into the set V: hence, this composition is such that $f(a) = f_\beta(f_\alpha(a))$. As a consequence, the value v assigned to an object a by f can be obtained by mapping a to its equivalence class $[a] = f_\alpha(a)$ and then such class to the value $v = f_\beta([a]) = f_\beta(f_\alpha(a))$ that is assigned, by means of f, to all objects belonging to the class itself.

The diagram in Figure 4 depicts this concept.

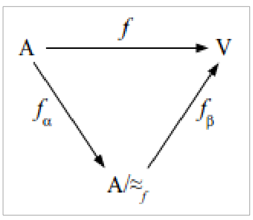

Figure 4. Basic algebraic structure of an evaluation process.

Under the mentioned assumption that f is intended to model an empirical evaluation process, the two functions f_α and f_β composing the function f can be meaningfully interpreted in terms of

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most determinate, but still universal, property of an individual object.

[17] As a consequence, this conceptual framework is compatible with both an *objective* standpoint, according to which properties and their kinds are intrinsic characteristics of the object to be evaluated, and an *operative* standpoint, according to which properties and their kinds are instead partially or totally determined by the evaluation processes that are concretely applied.

[18] Were uncertainty taken into account, the relation corresponding to the application of the function f would not be, in general, an equivalence, since its transitivity would not be guaranteed. This is a fundamental reason of the complexity of any formalization of uncertain evaluations.

components of the process itself:

- an *experimental* component, corresponding to f_α, that consists in the *determination* of the equivalence class which a given empirical object belongs to, as typically obtained by comparison among objects[19];

- a *representational* component, corresponding to f_β, that consists in the *assignment* of a value from V to the determined equivalence class.

The representational component f_β must preserve the information acquired by the experimental component f_α. This is guaranteed by the injectivity of f_β: distinct equivalence classes are represented by different values. Accordingly, the only constraint on the existence of f_β is on the cardinality of V, that must have enough elements to represent all equivalence classes: card(V) $\geq$ card($A/\approx_f$). On the other hand, the acquired information on $[a]$ is preserved not only by $f_\beta([a])$, but also by any function $\tau(f_\beta([a]))$, where τ:V$\rightarrow$V is a 1-1 transformation, i.e., a permutation of V. Values are then assigned "up to a group of transformations" G,[20] that can be called an *evaluation transformation group*. Evaluations f_i of objects in A can be then clustered in sets F=$\{f_i\}$, where:

- all f_i belonging to the same set F determine the same experimental component of the evaluation;

- the assignment of values performed by each f_i can be also obtained by any other mapping f_j belonging to the same set F by applying a suitable transformation τ in the transformation group associated to F.

Hence, a set F is uniquely determined by a transformation group G such that F = $\{f_i \bullet \tau \mid f_i \in F, \tau \in G\}$. The diagram in Figure 5 depicts this concept.

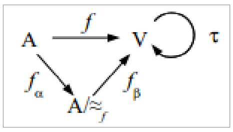

Figure 5. Evaluation as determined up to a transformation.

Each empirical evaluation process f can be conceived as the operative realization of a general property (such as shape or length), that can be indeed interpreted as a mapping from objects a to property values $f(a) = v$, as for shape(this_table) = rectangle. Accordingly, in the following we will sometimes adopt the overloaded denotation "the evaluation f" and "the general property f" for short, while acknowledging that the same general property can be evaluated by different processes, and

[19] It is precisely in this sense, we believe, that the VIM3 states that measurement is an experimental process [2.1], and that it "implies comparison of quantities" [2.1 N.2], an elliptical expression for "comparison of objects with respect to a general quantity". Indeed, measurement is based on comparison, as formalized by the function f_α, but it does not entirely consists of such comparison, as the function f_β must also be applied. Hence, comparison is necessary but not sufficient for measurement.

[20] As a typical example of transformation in the group G, consider the (scale) transformation due to a change of unit, e.g. from meter to inch, in the case of length. That the structure of the set of such transformations is a group can be easily shown: (i) the composition of any two transformations is a transformation (closure); (ii) the order in which any three transformations is applied does not change the resulting transformation (associativity); (iii) there exists the identity transformation (identity element); (iv) since transformations are bijective, for each of them an inverse transformation exists (inverse element).

therefore that any specific evaluation is generally only a partial counterpart of that general property.[21]

This interpretation leads to a better understanding of the equivalence relation ≈f. If what f evaluates is a general property, then such equivalence is in fact an f-related equivalence: the fact that two objects are mapped to the same equivalence class models that they "have the same property", as in a sentence such as "a_1 and a_2 have the same shape". Of course, this relation of sameness, or equivalence, does not apply to general properties, but to their instances to given objects. Indeed, the previous sentence implies that:

- the existence of the entities 'the shape of a_1' and 'the shape of a_2' is assumed, and
- such entities are indistinguishable with respect to the comparison process under consideration.

Following the VIM3 (note 1 to definition 1.1), the entities obtained by instancing a general property (such as shape) for a given object (such as the shape of this object) can be called *individual properties*. Accordingly, the equivalence relation $\approx_f$ stands for "having equivalent individual properties".

3.3. Measurement as evaluation process

The two components of an evaluation process, modeled by the mappings f_α and f_β, can be further analyzed as follows. The experimental component f_α is based on a comparison process aimed at determining the equivalence relation $\approx_f$. Such result can be obtained in different ways, due to the possibility of comparing objects:

- either relatively to the general property *f* (*direct* comparison, as in the case of two-pan balance) or to a different property, connected with *f* by means of a transduction effect that is possibly formalized by a law (*indirect* comparison, as in the case of dynamometer);
- either with each other (*internal* comparison, as in the case in which a two-pan balance or an uncalibrated dynamometer, whose scale has no marks, are used to compare two objects of unknown weigh to determine whether their weights are the same) or with given reference entities (*reference* comparison, as in the case a two-pan balance is used to compare an object of unknown weight and a standard, the standard itself operating as a reference object, or in the case an indication is obtained by applying an object of unknown weight to a calibrated dynamometer, the indication operating as a reference state).[22]

These options are synthesized in Table 1.

Table 1. Options about how a comparison process can be performed.

[21] This position can be characterized as a weakly operational one, contrasting to the strict operationalism that identifies properties and evaluation procedures: "we mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations" [24].

[22] In [25] these reference entities are characterized as the "selectable states" of the coupled system: object under measurement + measuring system. References can be indeed either objects or states: while in the first case the notation $a \approx_f r$ is correct, in the second one it must be understood as a shortcut. Just for the sake of simplicity we will assume that references are objects henceforth.

		Is a reference object involved in the comparison?	
		no	yes
Is the property transduced before comparison?	no	*direct internal comparison*	*direct reference comparison*
	yes	*indirect internal comparison*	*indirect reference comparison*

While a generic evaluation can be based on all four combinations of comparisons, measurement is performed as (direct or indirect) reference comparison (right-hand column options in Table 1), as it requires the measuring system to be calibrated with respect to a standard, whose property value is assumed to be known. Hence, in this case (we are still neglecting uncertainty) the process aimed at assigning the property value $v = f(a)$ can be understood as implying (1) the construction of a reference and (2) the application of the constructed reference, according to four steps:

1. Construction of a reference
   - Step 1.1: the selection of a set of reference objects is performed in four stages: (i) the definition of a set of reference objects R = $\{r_1, r_2, \ldots\}$ for the domain A; (ii) the definition of a comparison procedure such that for each object a in A one and only one reference r can be determined such that $a \approx_f r$; (iii) the definition of a way to produce sets of reference objects that fit the definition; (iv) the choice of a concrete set of reference objects. As a consequence of (i) and (ii), R is required to be *complete* (for all elements a in the set A at least one r exists in R such that $a \approx_f r$) and *minimal* (for all elements a in the set A at most one r exists in R such that $a \approx_f r$) with respect to A and the given reference comparison procedure[23].
   - Step 1.2: a property value v_i is associated with each r_i, under the condition that the mapping is injective, i.e., distinct references are mapped to distinct property values; this mapping can be modeled as a *representation* function f_R:R→V.
2. Application of the reference
   - Step 2.1: an object a is compared with the references in R and the reference r is determined such that $a \approx_f r$; the completeness and minimality of R guarantee that r exists and is unique; hence, this comparison can be modeled as a *comparison* function f_C:A→R, such that $f_C(a) = r$ if and only if $a \approx_f r$.
   - Step 2.2: the property value $v = f(a)$ is assigned such that $f(a) = f_R(f_C(a))$; the injectivity of f_R guarantees that distinct values are assigned to distinct objects.

The diagram in Figure 6 depicts this concept.

[23] The construction of a set of reference objects can be equivalently referred to as the construction of a set of reference properties of a given kind, namely the individual properties that characterize the objects. In the same vein, the comparison procedure can be viewed as a procedure that compares individual properties instead of objects. The present account is focused on objects instead of properties because (i) an individual property can only be identified as the property of a given object, (ii) properties can only be compared by comparing the objects they characterize, and (iii) the only way to specify how to produce a set of reference properties is to specify how to produce a set of reference objects instantiating them. Given the previous proviso, the concepts of reference property and reference object can be interchanged.

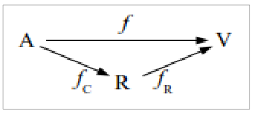

Figure 6. Basic structure of a measurement process: comparison + representation.

By merging the diagrams in Figures 4 and 6 we are led to a justification of the claim that measurement is a specific case of evaluation, as in Figure 7.

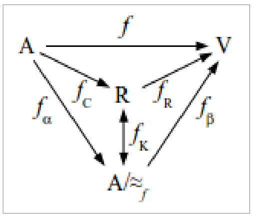

Figure 7. Structure of measurement as a specific case of evaluation.

Indeed:

- generic evaluation is (explicitly or implicitly) based on a comparison leading to map objects to equivalence classes, so that the property value $v=f(a)$ is obtained by $f(a)=f_\beta(f_\alpha(a))$ (outer triangle);
- measurement is instead specifically based on (direct or indirect) reference comparison, so that the property value $v=f(a)$ is obtained by $f(a)=f_R(f_C(a))$ (inner triangle).

The conditions of completeness and minimality on R imply that the *classification* function f_K:R→A/$\approx_f$ such that $f_K(r)=\{a\in A \mid f_C(a)=r\}$ is bijective. f_K models the operation of embedding each reference r into the equivalence class $f_K(r)$, so that r can be thought of as a "representative entity" for $f_K(r)$. This highlights that the determination of the equivalence of objects through references is a specialized version of the generic determination of their equivalence with respect to the general property f, i.e., by means of f_α.

3.4. A synthesis

Since *any* evaluation process, as far as it can be modeled by a function f, can be decomposed as $f(a)=f_\beta(f_\alpha(a))$, then:

evaluating =	comparing	+	representing
	(experimental component)	+	(representational component)

where the representational component is definable before performing any specific evaluation process and therefore is a priori[24], while the experimental component is, in the same sense, a posteriori. The conclusion is analogous in the case of reference comparison, where f_C and f_R are in fact specialized versions of f_α and f_β respectively. This can be interpreted in terms of the functional

[24] The representational component is a priori and still not completely conventional, since the choice of the target structure into which the domain is represented is constrained by the requisite of preserving the relations identified by comparison. As just mentioned, such component can be defined before performing any specific evaluation process but its definition is a critical part of the definition of the evaluation procedure.

relations between calibration and measurement:

- at first (= a priori) the measuring system is calibrated, i.e., the representational component is set up by identifying the reference associated with each realizable property value; this corresponds to the pointwise operative construction of the function f_R^{-1} (Steps 1.1 and 1.2 above);
- then measurement is performed, i.e., the experimental component is operated, thus determining $r=f_C(a)$, and its result is represented by means of calibration information, i.e., by inverting f_R^{-1} (Steps 2.1 and 2.2).[25]

This structure leads to some insights about the very concept of property and its multiple subtle meanings, and in particular it leads to the possible parallel interpretation of the entities in the diagram in Figure 7 as either objects or properties, as follows.

Any evaluation procedure requires the specification of an intended domain A by means of a given general property. The domain includes objects that in principle could be evaluated according to several evaluation procedures. The general property, e.g. *shape*, considered of an object corresponds to an individual property, such as *the shape of* the object a. For a given comparison procedure, the individual property of a can be indistinguishable from individual properties characterizing other objects in A. The equivalence class $[a]$ that is obtained by the application of the classification function f_K corresponds then to the one individual property, such as *a given shape*, that in principle can be possessed by different objects. Finally, any property value has the role of the representative of an individual property as obtained by f_R and $f_β$ respectively.

The diagrams in Figure 8 interpret the one in Figure 7 and highlight this parallelism.

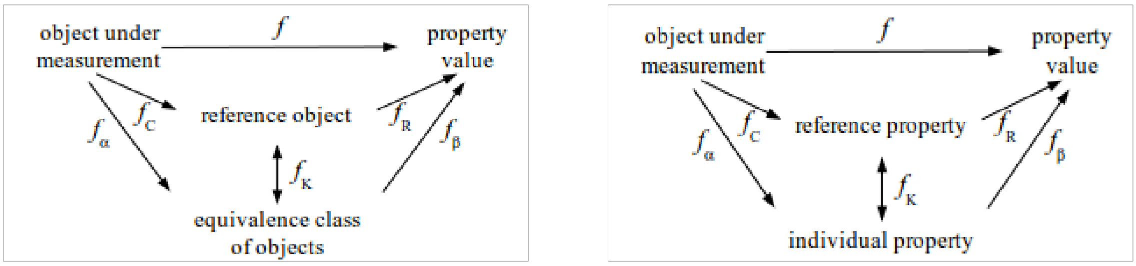

Figure 8. Interpretation of measurement in terms of objects (left) and properties (right).

In this sense, a property value, as an element of V, represents an individual property by identifying it within V. The basic formula of quantity calculus $Q = \{Q\}[Q]$ can be then reinterpreted: the quantity value $\{Q\}[Q]$ represents the individual quantity Q because $\{Q\}$ is the concept of a class

[25] This structure gives a simple explanation of the, somehow controversial, definition of 'calibration' introduced in the VIM3 (definition 2.39): "operation that, under specified conditions, in a first step, establishes a relation between the quantity values with measurement uncertainties provided by measurement standards and corresponding indications with associated measurement uncertainties and, in a second step, uses this information to establish a relation for obtaining a measurement result from an indication". As far as uncertainties are neglected, the first step corresponds to the construction of f_R^{-1} and the second one to the application of its inverse f_R. This also shows why "often, the first step alone in the above definition is perceived as being calibration" [2.39 Note 3]: the application of f_R is indeed the representational component of measurement, that is a priori with respect to the experimental component. In these terms, that measurement is modeled by f_C alone or by the combination of f_C and f_R is clearly a conventional issue.

within the classification given by (a subset of) the real numbers and completely specified by the unit $[Q]$. This can be immediately and consistently generalized to generic properties: the formula $P = \{P\}[P]$ states that the individual property P is represented by the property value $\{P\}[P]$, where $[P]$ is a given classification and $\{P\}$ is the concept of a class belonging to $[P]$ and therefore operating as a selector in it.

3.5. Example: galaxy classification

The results of the previous analysis can be exemplified by the procedure to classify galaxies on the basis of the Hubble sequence, i.e., according to a set of reference shapes matching the visual appearance of galaxies. A partial outline of the Hubble sequence is as in Figure 9:

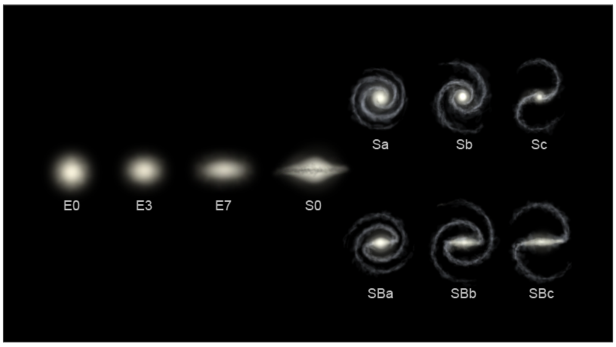

Figure 9. Partial outline of the Hubble sequence.

The Hubble sequence divides galaxies into three major groups based on their apparent shape, constituted by approximately elliptical (E), spiral (S), and barred spiral galaxies (SB) respectively. In spite of its descriptive basis, this scheme provides not only a classification but, within each group, also an ordering of the galaxies according to their relative shapes. For example, in the group E galaxies are ordered in virtue of the eccentricity of their shape. The definition of the Hubble sequence corresponds to the choice of the set of reference objects and their associated values, as presented in Steps 1 and 2 above, where the representation function f_R is defined extensionally by the one-to-one correspondence sketched in the partial outline of the Hubble sequence shown in Figure 9, being {E0,E3,E7,S0,Sa,Sb,Sc,SBa,SBb,SBc} the set V of property values.

On this basis, the classification and ordering process is based on the comparison between the apparent shape of a candidate galaxy a and the reference shapes in the sequence, such comparison process being simply specified as:

1) take the candidate galaxy a;

2) identify among the reference shapes in the Hubble sequence the one most similar to a.

Hence, the comparison function f_C induces both a partition on the domain A of observable galaxies (the equivalence relation being defined as $a_i \approx_f a_j$ if and only if the reference shapes of a_i and a_j are the same) and a total ordering within each group (the order relation for the group E being defined as

$a_i <_f a_j$ if and only if the reference shape of a_i is less eccentric than the one of a_j).

3.6. An ontological note

The concept of evaluation analyzed so far is critically based on the possibility to determine equivalence relations $\approx_f$. Hence, their ontological status is worth of some consideration. Four options can be taken into account.

- Option 1: relations $\approx_f$ are induced by evaluations f. Objects are mapped to property values independently of any a priori condition on the structure of A, so that in principle such structure could exist only a posteriori with respect to the evaluation: objects mapped to the same value are assumed to be equivalent for the evaluated general property. In synthesis: since $f(a_i) = f(a_j)$, then $a_i \approx_f a_j$.
- Option 2: relations $\approx_f$ are determinate independently of any evaluation f. Objects are assumed to be provided with individual properties independently of the application of any evaluation procedure: objects that are equivalent with respect to the given general property must be included in the same equivalence class and thus mapped to the same value. In synthesis: since ai $\approx_f a_j$, then $f(a_i) = f(a_j)$.
- Option 3: relations $\approx_f$ are induced by the experimental component f_α of evaluations. Objects are provided with individual properties by the application of the internal comparison procedure related to f_α: objects that are mapped to the same equivalence class (i.e., that are equivalent according to the given comparison procedure) are mapped to the same value. In synthesis: since $f_\alpha(a_i) = f_\alpha(a_j)$, then $f(a_i) = f(a_j)$.
- Option 4: relations $\approx_f$ are induced by the comparison function f_C of evaluations. Objects are provided with individual properties by the application of the reference comparison procedure related to f_C: objects that are mapped to the same reference entity (i.e., that are equivalent according to the given reference comparison procedure) are mapped to the same value. In synthesis: since $a_i \approx_f r$ and $a_j \approx_f r$, and therefore $f_C(a_i) = f_C(a_j)$, then $f(a_i) = f(a_j)$.

These options correspond to different overall interpretations of the diagram in Figure 7.

Options 1 and 2 focus on the mapping A→V, while neglecting the relevance of the whole structure. They basically imply inverse standpoints: Option 1 has basically no assumptions on A, but it is unable to justify why the values assigned to objects are to represent individual properties. On the other hand, Option 2 requires individual properties to be determinate as such, independently of any empirical process of interaction with objects. Options 3 and 4 acknowledge instead that empirical evaluations are structured procedures, generically based on comparison, as in Option 3, or specifically on reference comparison, as in Option 4. These options are synthesized in Table 2.

Table 2. Ontological options on the role of the evaluation.

Option	The evaluation:
1: since $f(a_i)=f(a_j)$, then $a_i \approx_f a_j$	*induces* the relations
2: since $a_i \approx_f a_j$, then $f(a_i)=f(a_j)$	*preserves* the relations
3: since $f_\alpha(a_i)=f_\alpha(a_j)$, then $f(a_i)=f(a_j)$	*determines* the relations by generic comparison
4: since $a_i \approx_f r$ and $a_i \approx_f r$ (then $f_C(a_i)=f_C(a_j)$), then $f(a_i)=f(a_j)$	*determines* the relations by reference comparison

The following analysis on PETs can be based on the ontologically less demanding Options 3 and 4, but is also consistent with Option 2. While Options 3 and 4 more clearly highlight the role that evaluation processes play in providing information about domains, Option 2 more intuitively understands the ontological grounds of relations $\approx_f$.

4. Property evaluation types

As discussed above, the evaluation f of a general property on the objects of a domain A implies determining the equivalence relation $\approx_f$, "having equivalent individual properties", on A. It is a fact that sometimes together with $\approx_f$ other relations are empirically meaningful on A. For example, an empirical relation of total order $<_f$, interpreted as "having an individual property less than another individual property", could be determined. In such cases, the further relations on A have to be in principle compatible with the equivalence relation $\approx_f$. In the case of $<_f$ this means that the empirical order is preserved among equivalence classes, i.e., for all a_i, a_j, $a_k \in$A, if $a_i \approx_f a_j$ and $a_i <_f a_k$ then $a_j <_f a_k$, thus making $\approx_f$ a congruence in the structure $\mathbf{A}$ = (A, $\approx_f$, $<_f$), that is determined by f on A. As further relations are possibly determined by the evaluation, a richer and richer algebraic structure is assumed on A and the set of property values V is correspondingly modified into a structure **V** to preserve the structural information that can be empirically acquired (we are adopting the usual notational convention of denoting a structure with a bold symbol, e.g., **A**, and its domains with the corresponding roman symbol, e.g., A).

The concept of *information consistency* for an evaluation f is fundamental here: a property value $f(a)$ must convey all and only the information empirically available on the structure **A**.[26] This is expressed by means of two complementary conditions. The evaluation f is consistently informative if and only if:

1) for each empirical relation in **A** there is a represented relation in **V** such that f preserves all relation instances; this guarantees that the information empirically acquired is maintained by the representational component of the evaluation;

2) **V** includes only relations that represent empirical relations in **A** in the sense of the first condition; this guarantees that the information conveyed by property values is actually representative of the information acquired by the experimental component of the evaluation.

[26] Of course, this concept of information consistency is empty if Option 1 in Section 3.6 is assumed.

For example, if $a_i <_f a_j$ (e.g., a_i is shorter than a_j) then f is consistently informative only if **V** includes an order relation < such that $f(a_i) < f(a_j)$ (the length value $f(a_i)$ assigned to a_i is less than the length value $f(a_j)$ assigned to a_j). Furthermore, under the assumption that the evaluation is only ordinal, f is consistently informative only if **V** includes the relation < but, in particular, not the operation +.

The two conditions of information consistency are easily specified in the case the evaluation procedure is based on reference comparison, and therefore specifically in the case of measurement. The critical role is played here by the reference set, that is assumed to be a structure **R** such that, e.g., $\mathbf{R} = (R, <_f)$. It is indeed the comparison function f_C that guarantees the information consistency, and induces the structure of R on A. As a consequence, in the specific case of measurement the empirical relations on A are not required to be directly observable, as they result from the following inference: for all $a_i, a_j \in A$, if $f_C(a_i) = r_i$ and $f_C(a_j) = r_j$ and $r_i <_f r_j$ then $a_i <_f a_j$.[27]

The mentioned conditions imply that any evaluation f is a homomorphism from **A** to **V**. In the example of (purely ordinal) lengths, in which f maps $(A, \approx_f, <_f)$ to the ordered set of non-negative real numbers), this is equivalent to the constraint that any f_β is an isomorphism from $\mathbf{A}/\approx_f = (A/\approx_f, <_f/\approx_f)$ into the ordered set of non-negative real numbers. These morphisms are invariant under transformations τ in the transformation group G, i.e., if f is a homomorphism, then also $f = f \bullet \tau$ is a homomorphism, and if f_β is a isomorphism, then also $f_\beta = f_\beta \bullet \tau$ is an isomorphism. This simply formalizes that facts such as a_i is shorter than a_j are independent of the assigned property values, and therefore in particular of the unit, as the property representation does not affect the instance $a_i <_f a_j$ of the empirical relation $<_f$. In other terms, the 'less than' empirical relation has to be invariant under the transformations $\tau \in G$.

The diagram in Figure 10 depicts this concept, and generalizes the diagram in Figure 7 by assuming that all graph nodes are relational systems and all arrows are morphisms.

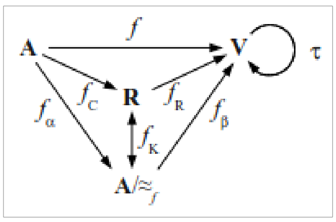

Figure 10. Algebraic structure of measurement as a specific case of evaluation.

Since not any transformation τ fulfills this invariance condition, G shall only include the *admissible transformations*, i.e., the transformations under which the relevant empirical relations are preserved.

[27] This is a major difference between the standpoint presented here and the already mentioned representational theories of measurement, that instead assume the direct observability of the empirical relations on A, and from it develop a concept of empirical meaningfulness ([26, 27]). From this point of view, our claim is that a more appropriate term would be "representational theories of evaluation", as they formalize unstructured morphic mappings (i.e., Option 2 in Section 3.6), not reference comparison (inner triangle in Figures 7 and 8, i.e., Option 4). This argument also justifies the apparent lack of interest that representational theories deserve to calibration.

Thus, for example, if $a_i <_f a_j$ then the guarantee that $f(a_i) < f(a_j)$ for all $f \in$ F is obtained by requiring that the transformations in G preserve the order $<_f$, and therefore that they are strictly monotonic. Hence, the structure of the group of transformations that are permissible according to this condition of invariance determines a classification on the set of property evaluations, such that *each class corresponds to a "type"*: a property evaluation type (PET).

4.1. The basic PETs

Let us consider the two extreme cases:

- on the one hand, f is unconstrained, i.e., the transformation group G contains all the injective mappings V→V. This is the case of evaluations performed by simply classifying the objects in A, as analyzed in Section 3, so that having the same individual property means only being equivalent according ≈f, i.e., belonging to the same ≈f-equivalence class. The corresponding type is customarily called the *nominal* (customary alternative terms are "classificatory" and "qualitative") type;
- on the other hand, f is unique, i.e., the only transformation contained in G is the identity. This is the case of evaluations performed by counting predefined objects, so that having the same individual property means having the same number of such objects. The corresponding type is customarily called the *absolute* type.

On this basis the types introduced in [16] can be schematically characterized as in Table 3.

Table 3. Basic PETs (adapted from [16]).

Type	Admissible transformations τ	Conditions of invariance	Features
nominal	injective transformations: if $v_1 \neq v_2$ then $\tau(v_1) \neq \tau(v_2)$	preservation of equivalence classes	classifications
ordinal	monotonic transformations: if $v_1 < v_2$ then $\tau(v_1) < \tau(v_2)$	preservation of order	orderings
interval	linear transformations: $\tau(v) = k_1 v + k_2$, $k_1 > 0$	preservation of ratios of distances: $(\tau(v_2) - \tau(v_1))/(v_2 - v_1) = k_1$	numerical mappings without a "natural zero" and a "natural unit"
ratio	similarity transformations: $\tau(v) = kv$, $k > 0$	preservation of ratios: $\tau(v)/v = k$	numerical mappings with a "natural zero" but without a "natural unit"
absolute	identity transformations: $\tau(v) = v$	preservation of values	numerical mappings with a "natural zero" and a "natural unit"

If read from top to bottom, this table lists more and more constrained types in their invariance conditions. If interpreted in reference to the same property, the transition from one type to the following, more constrained, one corresponds to an increase of information that is assumed to be conveyed by property values, on the basis of the requirement that such information is preserved by the evaluation, i.e., that the evaluation itself is consistently informative. For example, considering the case of temperature:

- a *nominal* evaluation leads to assess whether two objects belong to the same equivalence class or not, i.e., they have the same temperature or not; in this case, property values are only required to be distinct for objects with different temperature, a constraint that is fulfilled by any injective transformation;
- if the information is added on the empirical meaningfulness of an ordering among the equivalence classes, an *ordinal* evaluation is obtained, allowing to assess also whether the temperature of an object is greater than the temperature of another one; for such ordering to be expressed by property values, they must preserve the order, a constraint that is fulfilled by any strictly monotonic transformation;
- if the information is added on the empirical meaningfulness of a distance / difference among the equivalence classes, an *interval* evaluation is obtained, allowing to assess also how much the difference of the temperature of two objects is greater than the difference of the temperature of two other objects; this is the case of temperature evaluated in Celsius or Fahrenheit degrees, for which the value $(v_1-v_2)/(v_3- v_4)$ is invariant under scale transformations, as formalized by the linear transformations $\tau(v)=k_1v+k_2$;
- if the information is added on the empirical meaningfulness of a "zero" equivalence class, a *ratio* evaluation is obtained, allowing to assess also how much the temperature of an object is greater than the temperature of another one; this is the case of Kelvin and Rankine scales, for which the value v_1/v_2 is invariant under scale transformation, as formalized by the similarity transformations $\tau(v)=kv$;
- if the information is added on the empirical meaningfulness of a "unit" equivalence class, an *absolute* evaluation would be obtained: since no "natural" unit for temperatures has been defined yet (1 K can be hardly said to be "more natural" than, e.g., 1 R), the possibility of an absolute evaluation for temperatures is currently unknown / unavailable.

This sequence of transitions can be easily rephrased in terms of references instead of equivalence classes, thus adapting it to measurement (in this case the condition of nominal PET coincide with the requirements of completeness and minimality of references, as introduced in Section 3.3, so that by definition each measurement is at least nominal). This highlights once more that the concept of type unsurprisingly applies more to property evaluations than to properties "in themselves", whatever this could mean, since the same property can be evaluated by means of evaluations of different type. The sequence in Table 3 also shows that the PETs can be compared with each other in terms of the degree of structural information conveyed by property values, so that, e.g., the absolute type can be said *structurally richer* than the ratio one, and so on.[28] Furthermore, structural

[28] This relation of structural richness among PETs should not be confused with a relation based on the quantity of information (in the sense of the Shannon's theory of communication [28]) conveyed by property values, which is instead related to evaluation / measurement resolution and is independent of any structural requirements.

richness varies inversely to the size of the transformation group G associated with the evaluation, so that the type T_i is structurally richer than the type T_j when the group associated with T_i is included in the group associated to T_j.

4.2. A justification of the basic PETs

As presented in [16], the basic PETs are introduced on an ad hoc basis, and no formal justification is given of the implicit assumption that they cover all "interesting cases" of evaluation, and more specifically of measurement. Such a justification can be found, once more, in the relation that links the set $F = \{f_i\}$ and the transformation group G that specifies how the evaluations f_i can be transformed to each other, in terms of the following two concepts (see [29] and [27]).

- The set $F = \{f_i\}$ is defined to be m-point homogeneous, for m=1, 2, ..., if and only if for all pairs of strictly ordered sequences $v_1<...<v_m$ and $w_1<...<w_m$ of m elements of the target structure **V** there exists a transformation τ in the transformation group G such that $\tau(v_i) = w_i$ for i=1, ..., m. Furthermore, F is defined to be ∞-point homogeneous if it is m-point homogeneous for every natural number m. Then ratio evaluations are 1-point homogeneous but not 2-point homogeneous, interval evaluations are 1-point and 2-point homogeneous but not 3-point homogeneous, and ordinal and nominal evaluations are ∞-point homogeneous.
- The set $F = \{f_i\}$ is defined to be n-point unique, for n=1, 2, ..., if and only if, given any two functions $f_i, f_j \in F$, if f_i and f_j agree at n distinct elements of their image then $f_i = f_j$. Furthermore, F is defined to be ∞-point unique if it is n-point unique for every natural number n. Then absolute evaluations are 0-point unique, ratio evaluations are 1-point unique, interval evaluations are 2-point unique, and ordinal evaluations are ∞-point unique.

A PET can be then characterized by the maximum degree of homogeneity, m^*, and the minimum degree of uniqueness, n^*, of its set F. It can be shown that, under suitable conditions, $m^* \leq n^*$ and that if $0 < m^* \leq n^* < \infty$ then the only possible cases are $n^* = 1$ or $n^* = 2$, so that the only possible degrees are (1,1), (1,2), (2,2), (m^*,∞). Accordingly, it can be proved that a PET is:

1) ratio if and only if it is of degree (1,1);
2) interval if and only if it is of degree (2,2);
3) ordinal if and only if it is of degree (∞,∞);
4) nominal if the concepts of homogeneity and uniqueness do not apply.

This shows that basic PETs are classified by their degrees of homogeneity and uniqueness. Under rather general assumptions, it can be shown that the PETs listed in Table 3 are the most significant ones.

4.3. A metrological re-analysis of PETs

The characterization of PETs in terms of the invariance structure of the property value set V, as formalized by the transformation group G, is a consequence of the conceptual path followed by Stevens, aimed at encompassing in a single framework both "hard" and "soft" measurement, and therefore forced to the very generic definition: "assignment of numerals to objects and events according to rule" [16]. Stevens' interests were indeed focused on the connections between the rules for the assignment of numerals, the mathematical structure of the obtained scale types, and the statistical operations permissible with respect to such types, as presented in Table 4.

Table 4. PETs and permissible statistics (adapted from [16]).

Type	Permissible statistics
nominal	number of cases, mode, contingency correlation
ordinal	median, percentiles
interval	mean, standard deviation, rank-order correlation, product-moment correlation
ratio	coefficient of variation

Hence, since scale types are possible only because there is a certain isomorphism between the basic empirical operations performed on aspects of objects and the properties of the numeral series, Stevens concluded that scale types are dependent upon the specificity of the empirical operations performed on aspects of objects. In this sense, the possibility is open to differentiate between *property evaluation types*, connected to the set of admissible mathematical operation on the set V of property values, and *property evaluation procedures*, connected to the set of admissible empirical operation on the set of objects under evaluation. In particular, for evaluating a property according to a given type is not necessary that every operation that can be performed on V has an empirical correspondence in an operation performed on A [29].

Let us take into account two examples of ratio evaluations: absolute temperatures and masses. While their PET description is the same (both are modeled as numerical mappings that preserve ratios, with a "natural zero" and without a "natural unit"), such general properties exhibit a significant metrological difference: objects can be additively composed with respect to mass, but they cannot with respect to temperature (temperature is not so peculiar in this: the same applies, e.g., to density). As a consequence, a reference set for mass can be constructed by choosing a unit object and then iteratively applying the additive operation, i.e., juxtaposition in this case, whereas this strategy fails in the case of temperatures. The point is not that the type of either mass or

[29] We thank an anonymous referee for having directed our attention to this point by highlighting the need of a sharper distinction between the conditions that characterize an evaluation *type* and those characterizing an evaluation *procedure*. As a consequence, the conditions on V to be introduced to *characterize* a measurement scale are not to be confused with the conditions on A to *construct* the measurement scale. Hence, the question as to whether invariance conditions are sufficient to characterize measurement scales has to be answered positively, provided that measurement scales are interpreted as evaluation types and not as evaluation procedures.

temperature evaluation should be revised, but that it is unable to maintain this distinction between additive and non-additive mappings. Hence the issue arises: should PETs be further specified?

The evaluation type of thermodynamic temperature is ratio, despite of the lack of an additive operation, because a relation is known between thermodynamic temperature and a ratio-evaluated general property, length, such that temperature values are obtained from length values through a transduction effect and the proper calibration of the instrument that realizes such effect. While this example corresponds to the traditional, and well known, distinction between "extensive" and "intensive" quantities (that however remains hidden in the Stevens' framework), the result is clearly a general one, and it appears to be a feature of property evaluations: the same PET can include both homomorphisms and evaluations that only indirectly operate as homomorphisms. Some further examples are, for interval PET, calendar dates (direct, additive) and thermometric temperatures (indirect, non-additive), and, for ordinal PET, Mohs hardness (direct) and Beaufort wind scale (indirect).

As far as the metrological structure is taken into account, each PET can be split then into two sub-PETs, including respectively:

- *direct* evaluations, i.e., mappings that preserve the empirical relations / operations assumed to be applicable between objects in A, or between objects in A and references in R; if ρ_f is one of such empirical relations (here and in the following assumed binary and between objects in A just for simplicity of notation), and ρ is the corresponding relation in **V**, then a direct evaluation f is such that:

$$\text{if } \rho_f(a_1,a_2) \text{ then } \rho(f(a_1),f(a_2))$$

i.e., f homomorphically maps the objects in A to property values, such that, e.g., if a_2 is longer than a_1 then the length value assigned to a_2 must be greater than the one assigned to a_1;

- *indirect* evaluations, i.e., mappings obtained via another homomorphic mapping g; in this case the evaluation is based on a more complex, inferential structure, as follows:

$$\text{if } \rho_f(a_1,a_2) \text{ then } \sigma_g(a_1,a_2)$$

(e.g., if a_2 is warmer than a_1 then a_2 is longer than a_1, or an object obtained by transduction from a_2 is longer than an object obtained by transduction from a_1)

$$\text{then } \sigma(g(a_1),g(a_2))$$

(the length value assigned to a_2 must be greater than the length value assigned to a_1)

$$\text{then } \rho(f(a_1),f(a_2))$$

(the temperature value assigned to a_2 must be greater than the temperature value assigned to a_1).

For this inference to be correctly performed two conditions must hold:

1. an empirical relation is known between the general properties f and g, such that

$\rho_f(a_1,a_2) \rightarrow \sigma_g(a_1,a_2)$ is valid;

2. when represented by property values, such empirical relation can be inverted, i.e., $\sigma(g(a_1),g(a_2)) \rightarrow \rho(f(a_1),f(a_2))$ is valid.

The diagram in Figure 11 synthesizes the structure of this inference:

$$
\begin{tikzcd}
\rho_f(a_1,a_2) \arrow[r, dashed] \arrow[d] & \rho(f(a_1),f(a_2)) \\
\sigma_g(a_1,a_2) \arrow[r] & \sigma(g(a_1),g(a_2)) \arrow[u]
\end{tikzcd}
$$

Figure 11. Structure of an indirect evaluation.

Indirect evaluations can be applied in the case the relation $\rho_f(a_1,a_2)$ is not observed, or is even hypothetical. Let us take the example of $\rho_f(a_1,a_2)$ = the thermodynamic temperature of a_1 is double than the thermodynamic temperature of a_2. The empirical side of $f(a_1) = 2f(a_2)$ is $a_1 \approx_f (a_2 \circ_f a_3)$, where $a_2 \approx_f a_3$ and $\circ_f$ is a combination operation between objects in A. The fact is that no such combination operation $\circ_f$ is known. On the other hand, a relation between thermodynamic temperature and length is known (condition 1), as implemented in a suitable transducer, such that *were* $\rho_f(a_1,a_2)$ then $\sigma_g(a_1,a_2)$, e.g., a_1 would be six times longer than a_2 (thus assuming that the constant sensitivity of the transducer is 6/2 = 3 length unit / temperature unit).

On the basis of this example, let us compare the structure of the previous inference in the cases of thermodynamic (ratio PET) and thermometric (interval PET) temperatures:

thermodynamic temperature	*thermometric temperature*

temperature is transduced to length

a_1 is six times longer than a_2 (observation of $\sigma_g(a_1,a_2)$)

the length value of a_1 is six times greater than the length value of a_2 (representation as $\sigma(g(a_1),g(a_2))$

the temperature value of a_1 is six times greater than the temperature value of a_2
(representation as $\rho(f(a_1),f(a_2))$)

a_1 is six times warmer than a_2 (inference that $\rho_f(a_1,a_2)$)	no inference: interval PET does not preserve ratios

This confirms that the distinction between direct and indirect evaluations prevents identifying PETs with classes of morphisms but does not hinder the PET-based framework proposed here.

5. Conclusions: some consequences of the concept of PET

The type of a consistently informative evaluation f is derived from the structure of its range **V**, that in its turn is characterized by its associated transformation group. This leads to a straightforward interpretation of the traditional classification of concepts within the standpoint grounded on the

concept of PET.
- A concept is *qualitative* if its application to a given domain of objects implies the possibility of mapping them into a structure **V** that is simply a set, so that any injective transformation preserves the condition of information consistency for the mapping.
- A concept is *comparative* if its application to a given domain of objects implies the possibility of mapping them into a structure **V** that is an ordered set, so that any monotonic transformation preserves the condition of information consistency for the mapping.
- Finally, a concept is *quantitative* if its application to a given domain of objects implies the possibility of mapping them into a structure **V** that is an ordered set of numbers whose values, ratios, or interval ratios, are preserved by the transformations in the corresponding transformation group.

On the other hand, by introducing PETs the focus shifts from concepts to evaluation / measurement processes, and some generality and flexibility are gained. In particular, this standpoint allows taking into account the fact that the same general property can be evaluated by processes of different PET, where the change from one type to a structurally richer one can be interpreted as a structural upgrade of the evaluation, whereas the change from one type to a structurally poorer one corresponds to a structural downgrade of the evaluation itself.

Downgrading from a structurally richer than nominal type is always possible, as it is obtained by neglecting some structural information that in principle could be preserved by a suitable evaluation. For example, a length evaluation can be downgraded from ratio to interval type, by interpreting it as a distance evaluation, i.e., making the “zero” relative, but also to ordinal type, by assuming a ranking such as “long”, “average”, “short” for the set of values (as in the example presented in footnote 12). On the other hand, upgrading is more demanding, since it requires the introduction of appropriate structural information with respect to the previous evaluations. In particular, the upgrade from interval to ratio type implies a “natural zero” to be set, and the upgrade from ratio to absolute type implies a “natural unit” to be set. The possibility of upgrading could be obtained by scientific discoveries concerning empirical laws that connect different general properties and thus allow transferring PETs from general properties to general properties according to laws. Hence, for each general property the available knowledge in each historical period, both procedures and theories, establishes the structurally richest PET to which evaluations can be upgraded.

Such PET can be considered the type for the given general property.

As an example, temperature, traditionally deemed to be an ordinal property, was upgraded to an interval property with the introduction of the thermometric (e.g., Celsius and Fahrenheit) scales. Accordingly, while PETs are structurally related to evaluations and therefore are invariant characteristics of evaluations, *the classification of a general property under a given type is a time-dependent characteristic*[30].

The analysis can be now completed by proposing some comparisons between the standpoint based on the concept of PET ("PET theory" just for short) and the one assumed in the VIM3, and by discussing some consequences of the PET theory with respect to the characterization of the concepts of property evaluation, measurement, and property value.

5.2. The advantages of the concept of PET

The advantages of the PET theory relative to the standpoint assumed in the VIM3 are mainly connected to the following points.

1) *Focus of the concept to be defined*.

As already pointed out, the first difference relates to the very object to be characterized. Indeed, the PET theory focuses on the operative concept of property evaluation, and allows deriving a concept of property type as a specific, "upper bound", case. On the other hand, the VIM3 directly, although implicitly, attributes a type to subordinate concepts of 'property'. The greater generality and correctness of the PET theory do not seem in discussion here.

2) *Features of the definition*.

The PET theory unambiguously characterizes types in terms of structural invariance, and it is not forced to recur to any further undefined concept to introduce the PET-related concepts. The PET theory highlights how conventional is the threshold between "properties that are quantities", those having a magnitude according to the VIM3, and "properties that are not quantities". The distinction between quantities and non-quantities, that is sometimes even interpreted as a distinction between 'quantities' and 'qualities', is certainly controversial: in particular, ordinal properties have been traditionally considered non-quantities, as Maxwell did, under the Euclidean assumption that measurability implies evaluating ratios to units. The PET theory simply allows forgetting this distinction.

As a consequence, within the PET theory 'magnitude', a primitive concept in the VIM3, turns out to be definable in terms of PET: a property is a quantity (and thus has a magnitude according to the VIM3) if and only if it is an instance of a kind of property that can be evaluated by means of an evaluation process whose PET is at least ordinal.

3) *Taxonomy induced on the defined entities*.

With respect to the VIM3, the PET theory offers a better granularity, and in particular is able to distinguish among interval, ratio, and absolute properties / evaluations, that the VIM3 seems to

[30] Were further PETs taken into account, the relation based on structural richness could become a partial order (as it is in the case, in particular, of the log-interval type). Since in a partial order the upper bound is not guaranteed to be unique, a property could even have more than one type...

maintain undifferentiated within the undefined concept of quantity with unit. Furthermore, the basic relations among types in the PET theory are of inclusion, not of opposition as in the VIM3. For example, in the PET theory ordinal type is nominal type plus order, so that an ordinal property is a nominal property that is consistently informative on order, whereas according to the VIM3 an ordinal quantity is not a nominal property. Accordingly, the PET theory better accounts for the historical phenomenon of the evaluation upgrade and the operative phenomenon of the evaluation downgrade, and it clearly justifies why algebraically weak functions can be applied to structurally rich evaluations (e.g., in statistics, the median to ratio evaluations) and instead algebraically strong functions are not to be applied, in general, to structurally poor evaluations (e.g., the mean to ordinal evaluations). Hence, the diagrams in Figures 1 and 2 in the PET theory become as in Figure 12:

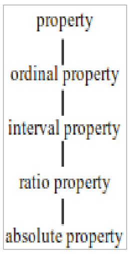

Figure 12. 'property' and some of its subordinate concepts according to the PET theory (inclusive version).

The generality of the PET theory allows easily adapting inclusive definitions to an exclusive version, were this considered appropriate. Figure 13 takes into account both inclusions (vertical relations) and oppositions (horizontal relations) between concepts (diagram at the left), where inclusions and oppositions are determined according to the PETs characterizing the related kinds of properties (diagram at the right):

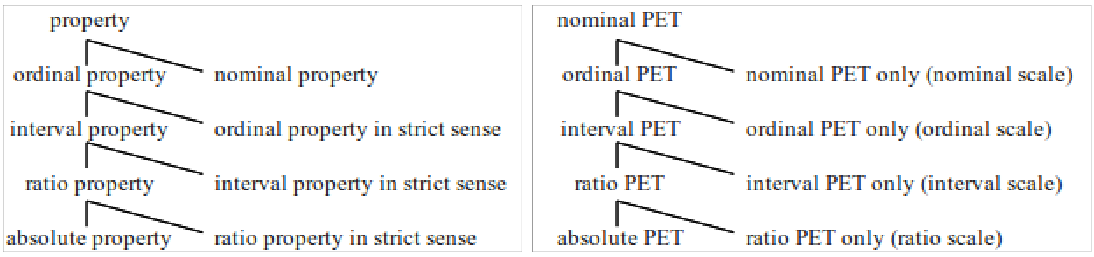

Figure 13. 'property' and some of its subordinate concepts according to the PET theory (exclusive version).

4. *Consequences for the concept of property evaluation*.

With respect to the Stevens' and the representational standpoint, the PET theory takes into account the structure of the evaluation, and not only its results: this allows distinguishing between direct and indirect evaluations, only the former being explicitly characterized as morphic mappings; furthermore, while the PET theory deals with generic evaluations, it can be specialized to measurement by emphasizing the role of references and therefore calibration.

5. *Consequences for the concept of measurement*.

The PET theory allows reinterpreting the very concept of measurement with respect to the VIM3 position, according to which a measurement is a "process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity". This definition is critically based on the two adverbs "experimentally" and "reasonably" that are undefined and unjustified: only [2.1 Note 2 – corrected version], "measurement implies comparison of quantities or counting of entities", gives some explanation on what this experimental activity could be. The PET theory justifies this implication and makes it more specific: the comparison of objects with respect to general properties is necessary for measurement, but not sufficient. Indeed, internal comparison is only able to convey the information that, e.g., two objects have the same length, but not which length is, and therefore which property value should be assigned to represent it. Measurement is instead based on reference comparison, thus highlighting why and how calibration is a requirement for measurement.

6. *Consequences for the concept of property value*.

The PET theory gives a general account of the concept of property value, operatively interpreted as selector within a classification, and easily allows to specialize it in the case of structurally rich PETs. In particular, in the case of ratio PET the reference set R embodies a concept of 'unit object' that, directly or indirectly, can be additively composed to generate all objects in R, so that the number of the repetitions, together with the unit, uniquely identify such objects. Accordingly, the Maxwell's formula:

$$Q = \{Q\}[Q]$$

is interpreted as a special case of:

$$P = \{P\}[P]$$

where P is an individual property and $\{P\}[P]$ a way of representing it by means of a given classification $[P]$ and a given class $\{P\}$ within it. In the case of additive properties, the classification remains usually implicit. Indeed, the formula $Q = \{Q\}[Q]$ can be interpreted both:

- *in an operative way*: the quantity Q of an object a is indistinguishable from the individual quantity generated by composing $\{Q\}$ times the reference individual quantity $[Q]$;

and:

- *in an abstract way*: the quantity Q of an object a is included in the class $\{Q\}$ within the classification provided by the real numbers once $[Q]$ is taken as the unit.

This interpretation mirrors the sense of the general formula $P = \{P\}[P]$.

7. *Consequences for the concept of measurability*.

Finally, in the context of the PET theory the assumption that only quantities are considered to be measurable (in [5.13], the VIM3 calls "examination" the evaluation of nominal properties) appears to be based on a conventional choice. The possible rephrasing of the definition of 'measurement', as "process of obtaining one or more property values that can be attributed to a property of an object by means of its experimental comparison to a reference", seems to be not less consistent than the VIM3 definition, and exemplifies the possibility of avoiding distinctions that are not grounded on conditions related to experimental facts.

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