preprint of a paper published in:
Measurement, 17, 2, 127-138, 1996, doi:10.1016/0263-2241(96)00022-X

The meaning of "quantity" in measurement

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Abstract

Measurement is aimed to assign a value to "a quantity of a thing". Therefore a clear statement of what a quantity is appears to be a required condition to interpret unambiguously the results of a measurement. However, the concept of quantity is seldom analyzed in detail in even the foundational works of metrology, and far too often quantities, are "defined" in terms of attributes, characteristics, qualities, etc. while leaving such terms in themselves undefined. The aim of this paper is to discuss the meaning of "quantity" (or, as it will be adopted here for the sake of generality, "attribute") as it is used in measurement also drawing several conclusions on the concept of measurement itself.

Keywords: *Measurement theory; Measurement science; Concept of quantity*

1. Introduction

To describe the world reference must be made not only to things, but also to their characteristics: such an assumption can be hardly escaped, being deeply rooted in the human way to know the world.

A thing can be identified by referring to it directly, by either explicit indication or a "proper name". Examples of the two cases are "move that thing", pronounced while indicating something, and "move *x*", where *x* is a proper name (that the communicants agree to associate with the same thing.) In such cases the reference does not convey any descriptive information to the thing referred to, but only allows its identification.

On the other hand, in many other situations descriptions are associated with things, in terms of their characteristics miscellaneously expressed as generic qualities ("move that red thing") or by means of more or less precise results of measurements.

However, what are such "characteristics"? In the scientific and technical field they are usually called "quantities", "magnitudes", "observables", "parameters", "dimensions", etc. Do these terms share some common meaning? And therefore *what is a measurand*?

Although usually commonly recognized, the problem is seldom dealt with in metrology and its solution is simply postponed as in (0), that defines a measurand as a «particular quantity subject to measurement», and a (measurable) quantity as the «attribute of a phenomenon, body or substance that may be distinguished qualitatively and determined quantitatively». This way, the question of meaning simply moves from "measurand" to "quantity" and from "quantity" to "attribute" while the problem remains unsolved.

If this question appears to be avoidable, however operatively marginal, in the case of traditional measurements it becomes fundamental when the characteristic being measured must be defined before it can be referred to. Such a definition may be inherently complex if, for example, we are considering the characteristic of the "quality" of the product of an industrial process or the "pain" of a patient.

In this paper, the concept of quantity will be analyzed in its central role for measurement. Among the consequences of the analysis and the following modelization:

* a definition of measurement reflecting the proposed concept of quantity will be suggested.
* according to the representational point of view to a theory of measurement, a measurement is formalized as a morphic mapping (0); but should we use either an isomorphism or a homomorphism? This matter will be considered, and an answer given;
* finally, several points arising in this paper can be considered as general guidelines in the definition of "new" quantities.

Following the usage of the term "attribute" given in the *International vocabulary of metrology*, the term "attribute" will be used in a general sense to denote the concept under analysis: this assumption makes clear that we will not restrict a priori our analysis to characteristics with quantitative / numerical values, but we will also allow the concept of measurability to be used in discussing cases that involve for examples purely ordinal characteristics (this generalization is a well-known result of the representational point of view to a theory of measurement).

A final introductory point must be mentioned referring to attributes. Often misinterpreted in their conceptual nature, it should be clear that attributes are not (material) "things" but mental constructions that we adopt to describe things. This ambiguity sometimes emerges in scientific and technical terminology[i]. On the contrary; our basic assumption is that measured things and their measurands *have a different nature*.

2. Background: descriptions, attributes and values

In their description, things are characterized by means of linguistic terms. When a thing is described in terms of attributes it is characterized by means of linguistic terms considered as the values of such attributes. In this sense, the length of a thing is an attribute whose values can be chosen among the elements of, e.g., a suitable subset of the positive real numbers but also the set {"long", "medium", "short"}.

In dealing with a foundational topic such as this, some terminological problem**s** can hardly be avoided. Terms such as "thing", "attribute", "property", and "description" are used here with a specific meaning that must be explained in order to understand the following discussion and to allow the reader to focus his attention on the concepts analyzed instead of on the terms adopted to denote them.

Let us consider the following:

S1="the height of that man is 1.95 m"

S2="that man is very tall"

In both S1 and S2 something ("that man") is described by means of a characteristic (the height) evaluated on it. We will use the term *thing* to denote the "something" that is described in a sentence as S1 or S2. Indeed, according to the Oxford Dictionary, a thing is «a material or non-material entity, idea, action, etc., that is or may be thought about or perceived». The term "thing" therefore seems preferable to, for example, the more restrictive "object", that denotes «a material thing that can be seen or touched» and thus would incorrectly suggest the interpretation that only material things can be described and measured.

S1 and S2 differ in the way the description is expressed. In the first sentence the characteristic is explicitly stated with its value ("1.95 m"). In the second sentence, it is the term "very tall" that brings implicit information also on the described characteristic.

We will use the term *attribute* to denote a characteristic of the thing described that is explicitly expressed and whose value is explicitly stated in the sentence, as in S1. We use the term *property* to denote a characteristic of the thing described appearing in a linguistic construction as in S2 (as will be discussed in the following, a more formal distinction between properties and attributes is that an attribute assumes one value chosen in a given set while the assertion of a property can be either true or false, i.e., a property assumes a Boolean truth value). Therefore in S1, "height" is an attribute of "that man" with "1.95 m" as the value, and in S2, "very tall" is a property of "that man".

In its more general form, a *description* can be defined as a triple u, A, v where:

* the term u is a proper name denoting the thing which the description refers to. It should be assumed that such a designation unambiguously identifies the thing whose name is u. Note that we are using the term "thing" in a very general sense to denote not only material things but also phenomena, processes, events, etc.;
* the term A denotes an *attribute* that is assumed that can be evaluated for u (to be precise, we should say that the attribute can be evaluated *on the thing* denoted by u. On the other hand, as customary, such a distinction will be omitted, leaving it to the reader to understand when the reference is to things and when to their names). The set of things to which the attribute can be applied will be called the *domain* of the attribute and denoted as Dom(A). Dom(A) can be thought of as a suitable subset of a set of things. Indeed, it is an empirical fact that not every attribute can be applied to every thing. There are many examples of such a non applicability: dimensional attributes relating to the spatial extension of bodies do not apply to gases or liquids; several macroscopic attributes such as color, pressure, etc. do not apply to microscopic particles; typically, social attributes do not apply in physics and vice versa;
* the term v denotes a *value* for A. v is expressed by means of the elements of a set SYMB called the universe set for A.

If one wishes to express *the way* the description is obtained, a fourth term must be specified:

* an *operation op*, such that v is the result of the application of *op* to u. The set of things to which the operation can be applied will be called the domain of the operation and denoted as Dom(*op*). For a given description of u, A, and v, if *op* is the operation used to evaluate A, then it can be assumed that $\mathrm{Dom}(op) \subseteq \mathrm{Dom}(A)$. In other words any operation evaluates a single attribute, whereas an attribute can in principle be evaluated by more than one operation.

We will write a description with the functional notation $A(u)=v$ or $A_{op}(u)=v$, or $A_{op}(u,t)=v$ to make explicit the time t in which the description has been obtained. This formalization makes the elements of a description explicit but leaves the basic issue of "what an attribute is?" unanswered and requires further analysis on the relations among the terms in $A_{op}(u)=v$. What guarantees that a thing u can be described by means of an attribute A, i.e., does u belong to its domain, Dom(A)? And vice versa: what guarantees that A can be evaluated for u? Which relation should be assumed between an attribute A and an operation *op* when A is evaluated by *op*?

3. Some features of attributes

3.1. Qualitative and quantitative attributes

According to traditional terminology, attributes can be distinguished as "qualitative" and "quantitative". This distinction is related to the hypothesis that knowledge of a given domain follows an evolutionary process so when the domain is initially investigated, the knowledge acquired only allows things to be classified on the basis of certain identified characteristic which express analogies and differences among things. In a second phase, knowledge could become more specific allowing an order to be defined among things according to different degrees of intensity shown by the characteristic. Finally, knowledge could proceed to a point in which the characteristic can be meaningfully evaluated in quantitative terms so that things are not only given an order but also a metric, i.e., a distance from each other.

An attribute is said to be "qualitative" whenever it allows things to be classified or ordered. It is "quantitative" if its evaluation induces a metric on its domain (a characterization of this kind is presented in Philosophy of Science, e.g., in (0), Genetic Psychology, e.g., in (0), also discussed below, Metrology, e.g., in (0)).

Clearly, such an evolution from quality to quantity is not necessary. For example, in many cases biologists are only able to classify but not to define meaningful criteria to order the things they are examining (in this context the work of the Swedish naturalist Carl Linneus, who, in 1735 published a system of botanical classification is usually mentioned). It should be noted that here we are using the concept of "meaningfulness" in a technical meaning, as defined, e.g., in (0).

On the other hand, the analysis that has been mainly carried out in this century on the concepts of theory and model and the formalization of the algebraic structures have highlighted that a set of given cardinality is characterized not by the "names" of its elements but by its structure, i.e., the set of relations defined on it. In this sense, it is not relevant whether the universe set for an attribute is SYMB={1,2,3} or ={a,b,c} if the same relations are defined in the two cases, i.e., if an isomorphism can be established between the two sets (a couple $<X, R_X>$, where X is a set and R_X a set of relations on X, is defined a "relational system". Therefore an attribute is formally characterized (but isomorphisms) by a relational system, and not a set).

A plausible distinction between attributes with qualitative and quantitative values should be founded on a corresponding distinction among "types of relations". The traditional vision that considers measurement as the operation allowing to express quantitatively an attribute identified qualitatively (where "quantitatively" usually means "by numbers", on the basis of the dubious assumption that the concept of number is well and uniquely defined) appears quite naive. The representational point of view to a theory of measurement (0) actually formalizes the concept of types of relations, expressing it in terms of measurement scales and their types (e.g., nominal, ordinal, interval, …), but it is unclear if such a point of view agrees with the traditional approach (cf. for example (0)), viz., assumes ordinal measurements as qualitative and interval ones as quantitative, or models empirical relations among things as always qualitative and formal relations among symbols as always quantitative.

However, due to its generality this topic will not be developed any further.

3.2. Singular and non-singular descriptions

Once the universe set SYMB and its algebraic structure have been chosen, the form of the values that the attribute can assume must still defined. Indeed, the fact that a description $A(u)=v$ specifies *one* value, v, does not imply that such a value be a *single* element of SYMB.

We will call $A(u)=v$ a *singular description* with respect to a given set SYMB if $v \in$SYMB and a *non-singular description* in all other cases. In a non-singular description, v could be an interval or a generic subset of SYMB or even a more complex entity such as a probability distribution on SYMB, a fuzzy subset of SYMB, etc.

To discuss the meaning of non-singular descriptions, the simple case in which v is a subset of SYMB can be considered. Here the non-singular description corresponds to the disjunction of several singular descriptions, one for each element of SYMB in v. For example, given SYMB={1,2,...,6}, the non-singular description $A(u)=\{2,4,6\}$, specifying the value "even number", is equivalent to:

$A(u)=2$ OR $A(u)=4$ OR $A(u)=6$.

The dependency of being singular or not on the universe set should be noted: with respect to SYMB={"odd number", "even number"} the previous description would be singular.

The choice to adopt singular or non-singular descriptions is pragmatic: if, on one hand, values in singular descriptions are easier to handle from the formal point of view, on the other hand non-singular descriptions can be used to make explicit the presence of some inexactness in the evaluation, a characteristic that is specifically relevant to measurement.

3.3. The origin of attributes

The idea that a thing can be described by means of the values of some attributes must be plausibly preceded by the intuition of some sort of conservation for such values. In other words, the thing must show some stability with respect to the attribute before one is able to recognize that the attribute itself can be evaluated on the thing.

In the studies on the development of the concept of quantity in the child (see, e.g.,(0)), it has been shown for example that the notions of conservation of weight and volume for deformable physical objects emerge only relatively late (when children are between 8 to 12) and are preceded by the knowledge that things have a property of "conservation of substance". A child observing the deformation of an object reaches the conclusion that the "quantity of matter" does not change during the operation before he is able to understand that also its weight and volume are constant.

Therefore, at the origin of the possibility to use any attribute there exists a qualitative knowledge on the things. Even a purely logical analysis seems to support this conclusion as paradoxically confirmed by the "more quantitative" attributes: those relating to counting. Indeed, one of the operative problems at the basis of any counting refers to the selection, within a set, of the things to be counted. Before the counting can be done, a qualitative (yes-no, count-not count) problem of identification must be solved. Indirectly, this corroborates the hypothesis that the knowledge of things through their attributes follows an evolutionary process from qualitative to quantitative, and structurally more complex, attributes.

Such a process is significant in relation to psychology and therefore to the history of each individual, but can also be interpreted in terms of social history. From this point of view it can be noted that things have been described by means of attributes prior to any formalization of the concept of attribute. This led to using some attributes in a way that would be considered rather improper according to our current criteria, for example, with respect to the choice of the units of measurement. This is the case of some dimensional attributes such as those referring to the walking distance between places, traditionally evaluated in terms of units of time, typically walking time (see (0)).

These examples highlight the importance of the pragmatic component in the definition of the attributes: why should the distance between two towns be defined in units of length if the purpose of such an evaluation is to establish the time needed to reach one town from the other?

4. The definition of attributes

4.1. Introduction

Traditionally, it is considered that the meaning of a term can be defined according to two different strategies that can be called "intensional" and "extensional" respectively.

The intensional meaning refers to the "internal content" of the concept denoted by the term as it could be found for example in a dictionary. Therefore, the intensional meaning of a term is explained using words whose meaning, assumed to be already known, is considered equivalent to that of the term. On the other hand, the extensional meaning of a term is given when the things that one would accept to denote that term have been listed.

A suitable way to define a meaning for the terms of attribute**s** appearing in the descriptions (i.e., the terms A in $A(u)=v$) is needed to assure that the results of measurement expressed by such descriptions are well-defined and unambiguous.

Is there any intensional meaning for the terms of attribute**s** that is useful for measurement purposes? We do not believe so and agree with the well-known critique of Bridgman (0) asserting that the fact that «many concepts of physics have been defined in terms of their properties» prevented a suitable analysis of their meaning. Indeed, in a more or less direct way, an intensional meaning refers to the same term it is defining, and therefore it is ultimately a tautology.

In this sense, to know that for example "length" means "extent from end to end" (one of its possible intensional meanings) does not really increase our understanding of how to interpret the results of the measurement of length, because if one tries to explain the meaning of the terms adopted in such a definition one will be forced to use the defined term.

While for a term such as "length" the issue seems unimportant because of the common concept, more attention should be paid to the meaning of terms such as those previously mentioned, the "quality" of the product of an industrial process or the "pain" of a patient.

In this Section the extensional meaning of the attributes will firstly be discussed. Later a third strategy, called "operational", to define the meaning of the terms of attribute will be presented, and its merits over the previous two will be highlighted.

4.2. The extensional meaning of the attributes

As previously considered, the assumption lying at the basis of the possibility to refer in linguistic terms to things by means of descriptions is that things "have characteristics", or, as we state here, “show (or possess) properties”, i.e., “qualities” allowing things to be characterized and distinguished among themselves.

A typical proposition asserting that a thing shows a property is “<thing> is <property>“, where the term <*x*> denotes a variable. Therefore “this table is white” is an example of such a general form, where <thing>=“this table” and <property>=“white”. Note that <property> can be arbitrarily complex from a linguistic point of view and could even specify a numerical term.

Propositions of this kind bring a descriptive information on <thing> only if a truth value is associated with them so, asserting that “<thing> is <property>“ means that “not-(<thing> is <property>)“ is in principle possible, and actually false.

This concept of property can be formally expressed by means of a unary predicate P, such that $P(x)$ is read as “the thing x, not specified, shows the property P”. Since the formula $P(x)$ contains the occurrence of a free variable, it does not have a definite truth value: generally, for at least one x x is actually P but for some other**s** no. To become true or false, the formula must be closed. This can be done in two basic ways. Firstly, by adding a quantifier for x, i.e., $\forall x P(x)$ ($P(x)$ holds for all x's in the domain) or $\exists x P(x)$ (there exists in the domain at least one x such that $P(x)$). Secondly, by substituting the variable x with a constant term u interpreted on the domain of things, i.e., $P(u)$.

This second form allows a specific fact to be asserted: a given thing u is P, as may be obtained by measurement.

To establish the actual truth or falsehood of a proposition $P(u)$ is not a formal but a semantic matter, conditioned by the meaning attributed to the property P: such a meaning depends on the context. Therefore, that a given thing u is white, or tall, or light, depends on the set of things under consideration. Moreover, the meaning of P could be recognized as vague, or ambiguous, and thus the truth value of $P(u)$ could be neither completely true nor completely false but intermediate or undetermined.

To suitably express the results of measurements, more semantically specific forms are needed to avoid, or to limit, the problems of context dependency, vagueness, and ambiguity of the meaning of properties.

In many cases the description of a thing in terms of its properties is expressed by more complex forms than “<thing> is <property>“.

Let P_1, …, P_n be an ordered set of properties such that for each thing u in the domain it is empirically verified that there exists a unique index i, $1 \leq i \leq n$, such that $P_i(u)$ is interpreted as true in the domain, $P_j(u)$ being false for all $j \neq i$. The properties of such a set are then exhaustive and mutually exclusive.

In this situation, the information that the assertion $P_i(u)$ brings is twofold: the thing u shows the property P_i, and does not show any other property P_j. On the other hand, the above-mentioned predicative form does not make this explicit: to assert that “this table is white” is not sufficient in principle to exclude that the table is also red (in the same sense in which it is not excluded that it is tall or light), at least whenever it is not made explicit that such properties are mutually incompatible.

In the case of a set of exhaustive and mutually exclusive properties, an alternative form to $P(x)$ can be adopted, based on the extension to the concept of predicate that Carnap (0) defines “functor”. If $\underline{P}$ is an identifier for such a set of properties, the expression $\underline{P}(u)=i$ indicates that the thing u shows the i-th property of the set $\underline{P}$.

If, as previously considered, the form $P(u)$ is expressed in a natural language as "u is P", how can $\underline{P}(u)=i$ be expressed?
The plausible answer is that a functor defines an attribute, and the form $\underline{P}(u)=i$ can be read as "<attribute> of <thing> is <value>" as in the case "the color of this table is white", properties here being values for attributes (here "property" therefore means "value for an attribute").
In these terms, an attribute is the name for a set of values considered as exhaustive and mutually exclusive (cf. (0) where an analogous vision is presented). This leads to the so-called "extensional concept" of an attribute whose meaning is expressed by the extension of a set, i.e., the explicit listing of all its elements. Therefore "what does 'color' mean?" would get an answer of the kind "it is a name for the set {"white", "red", "yellow", etc.}".
Besides "<thing> is <property>" and "<attribute> of <thing> is <value>", there exists the third linguistic form; "<thing> has <attribute>" (as in "this table has a color"), that is worth mentioning.[ii]
"<Thing> has <attribute>" brings the information that the attribute-functor can be applied to the thing although a value is not yet specified. The forms "<thing> has <attribute>" and "<attribute> of <thing> is <value>" differ with respect to their empirical contents, the former conceptually preceding the latter. Indeed, the evaluation of an attribute of a thing can be done only after one recognizes of being able to evaluate such an attribute with respect to that thing (cf. (0) and (0)). Attributes are then characterized not only by a set of values but also by a "domain". This position extends a purely extensional approach to the concept of attribute towards operationalism and therefore will be considered again in the following Section.
A further comparison of the use of the two forms asserting that a thing shows a property – "<thing> is <property>" and "<attribute> of <thing> is <value>" – seems to be interesting. The former is more direct and conceptually simpler since it does not require the explicit indication of an attribute. However, for the same reason it is necessary for the property to be expressed in such a way as to bring information not only on a value but also implicitly on the set it belongs to. That is the reason for which "this table is white" is allowed, but not "this table is three" and "this table is three meters".
Only "self qualifying" terms can be used in this case and natural languages exhibit several forms of this kind, often in dichotomic, positive-negative pairs (such as long-short, far-near, heavy-light, strong-weak, hot-cold, etc.). Moreover, such terms are usually adopted with linguistic modifiers "very", "enough", etc. On the contrary, the attributes specifically introduced in the scientific and technical field generally do not admit this specific linguistic qualification, plausibly because of their use based on numerical evaluations. For example, if the attribute "length" has the qualifiers "long" (positive) and "short" (negative), what about "power", "angular momentum", "entropy", etc.? The conclusion is that the form "<thing> is <property>" cannot be used in these cases: while "this table is one meter long" and "the length of this table is one meter" both exist and are substantially equivalent, the proposition, "this wire 500 ohm resistant" would be considered at least unusual.
Finally, this justifies the well-known fact that the results of measurements, requiring an explicit indication of the measured attribute, are expressed as "<attribute> of <thing> is <value>", i.e., $\underline{P}(u)=i$.
The extensional approach gives some meaningful contribution to the question, "What does 'attribute' mean?" but does not provide a full solution to it. The answer cannot be a purely

formal one and a complementary approach, also taking into account the empirical component, must be considered.

4.3. The operational meaning of attributes

To a question of meaning posed by Alice the answer was «the best way to explain it is to do it» (0). In the same spirit (we presume), but in a more academic manner, Bridgman (0) considered that «the concept of length implies the group of operations by which length is determined». And even more generally, «the concept is a synonym of the corresponding group of operations». So, «an operative definition of "length" could specify a procedure involving the use of rigid rods to determine the distance between two points» (0) (it is interesting that in *Flatland - A Romance in many Dimensions* (0), almost written contemporarily as the quoted *Alice in Wonderland*, the "flat", two dimensional men assert the impossibility of a third dimension precisely because of their impossibility to measure it).

This approach, thus founded on the refusal to idealize the concepts to be applied in the scientific and technical fields and on the emphasis for the need for their empirical reference, has been defined as "operationalism" (as originally proposed, «the real meaning of a term must be found by examining how a man uses it and not what he says about it» (0), operationalism seems to agree with the original formulation of the neo-positivistic point of view, historically contemporary, «the meaning of a proposition is the method of its verification» (0)).

With respect to the distinction between operative and theoretical terms (cf., for example, (0) and (0)), the most radical interpretation of operationalism would lead to the refusal of any meaningfulness to the latter and such a flaw has been recognized even by the original proposer of the approach, P.W.Bridgman, e.g., in (0).

On the other hand, for the purpose of this work we can consider a restriction of the operational assumption: "the *attribute* is a synonym of the corresponding group of operations", or even more specifically: "the attribute is a synonym of the corresponding measurement system", analogous to: «every measurement system defines an observable» (0). In this view, each term of attribute becomes the "proper name" that denotes a specific measurement procedure, and in a description, $A_{op}(u)=v$, the meaning of the attribute A is then completely specified by the operation *op* so that the notation A_{op} is redundant.

But, if it is actually the case that each group of operations identifies a different attribute, the usual terms of attribute would lose any reference capability. For example, instead of "length" one should say "attribute evaluated with the method x", and since a "length" (in the usual meaning of the term) can be measured with different methods, this view would force one to admit that each of these methods actually measures a *different* attribute.

Such a radically anti-idealist position can hardly be maintained when the effectiveness of the descriptions based on models implying the use of general terms is recognized (a fanciful tale of the consequences of avoiding any use of general terms is in (0)). For example, a huge part of the *corpus* of the physical sciences is expressed in the form of laws stating relations among values of attributes considered as general terms, independently of the specific method adopted for their measurement. As A.Pap considers in (0): «unless one is justified in regarding the mass of the earth, the mass of an electron, and the mass of a football as determinate forms of one and the same determinable property, one cannot justify extrapolation of numerical laws for the purpose of calculating values of physical variables which are not accessible to measurement».

At the opposite extreme, the validity of the antithetical position can also be supported: «I assert that if the introduction of a quantity promotes clarity of thought, then, even if today we have no means to determine it with precision, its introduction is not only legitimate but also desirable. What is not measurable today can be measurable tomorrow» (J.J.Thomson, quoted in (0)).

The need of a mediation is then compelling. Indeed, while «the operative definition of a physical quantity is given by the description of the sequence of the operations needed to measure such a quantity, … in general there can be different sequences of operations that define the same physical quantity» (translated from (0)).

In this respect, it has been considered that «the fact of identifying two attributes measured with different operations requires the formulation of a hypothesis to be confirmed by means of accurate experimentation» (0).

Indeed, given two descriptions $A'_{op1}(u,t)=v_1$ and $A''_{op2}(u,t)=v_2$ of the same thing u and for the same time t, according to the radical operational point of view, from $op1 \neq op2$ it follows that $A' \neq A''$. On the other hand, let us suppose that on the basis of "accurate experimentation" suitably set up in a meaningful set of situations obtained varying the thing u under observation and the time t of observation is always $v_1=v_2$ (such an equality between values of attributes should be considered but a scale transformation (i.e., it is an equality of symbols having a unit of measurement). Moreover, it is clear that in presence of inexactness, equality does not necessarily mean identity of symbols. However, this point will not be investigated further here). This result could be inductively considered as a confirmation of the hypothesis of *operative indistinguishability* for the attributes A_1 and A_2. In such a case one could write both $A_{op1}(u)=v$ and $A_{op2}(u)=v$ to denote that a description referring to the same attribute A has been obtained with two different operations, *op1* and *op2* (on this topic see also (0) and (0)).

Therefore, in this *generalized operational view,* each term of attribute is considered to designate a "cluster concept" whose components have a meaning that is partially overlapped but not necessarily fully coincident so each operative procedure defines an attribute only partially (cf. also (0) where this idea is further discussed. In analogous terms, Carnap asserts that «instead of saying that we have many concepts of length, each of them defined by a different operational procedure, I prefer to say that we have a single concept of length, that is partially defined by the whole set of physics including the rules for all the operational procedures used for the measurement of length» (0)). In this sense, each method to measure a "length" actually identifies only a sub-concept of length and the whole meaning of the concept emerges as the aggregation of a suitable cluster of such sub-concepts. The complexity of such whole meanings is then apparent, since the constituting sub-concepts highlight only parts of the entire frame and typically evolve with the time (consider, e.g., the concept of "mass", whose meaning, undoubtedly in the past including the hypothesis of its conservation under certain transformations, was forced to change because of the Einstein's theory of relativity).[iii]

The conditions justifying the hypothesis that two concepts are the specialization of a single, more general concept are as follows: let op_1 and op_2 be two operations leading to the evaluation of the *a priori* distinct attributes A_1 and A_2. If $\mathrm{Dom}(op_1) \cap \mathrm{Dom}(op_2) \neq \emptyset$, i.e., there exists a non-empty set of things such that both operations can be applied to them, and if, for all $u \in \mathrm{Dom}(op_1) \cap \mathrm{Dom}(op_2)$ and for all t (i.e., varying the experimental situation), $A_1(u,t)=A_2(u,t)$, then the hypothesis that op_1 and op_2 lead to the evaluation of the same attribute is corroborated.

Let us call R_1 such a relation on the set of the operations: if $R_1(op_1,op_2)$ then op_1 and op_2 *can* be adopted to evaluate a same attribute. In this case, we say that op_1 and op_2 are *directly related* with each other in the definition of such an attribute.

It is easy to show that R_1 is reflexive and symmetrical but not transitive, i.e., R_1 is a compatibility (sometimes also called tolerance) relation: to formalize the relation of "evaluation of a same attribute" among operations, an extension of R_1 is required.

Then, let R_2 be a relation such that $R_2(op_1,op_2)$ if and only if $R_1(op_1,op_2)$ or there exists a sequence $\{op'_1,...,op'_n\}$, $n \geq 1$, such that $R_1(op_1,op'_1)$, …, $R_1(op'_{i-1},op'_i)$, …, $R_1(op'_n,op_2)$.

If $R_2(op_1,op_2)$, but not $R_1(op_1,op_2)$, then we say that op_1 and op_2 are *indirectly related* with each other, via the sequence $\{op'_1,...,op'_n\}$, in the definition of an attribute.

R_2 is an equivalence relation and is such that $R_2(op_1,op_2)$ iff by means of op_1 and op_2 is evaluated a same attribute, in the above specified sense. Therefore we can conclude that *the attribute is synonym of the corresponding R_2-equivalence class of operations*. It should be noted that a R_2-equivalence class of operations can only be defined by induction since no empirical activity can be performed on *all* things and for *all* times. In this sense, the definition of a R_2-equivalence class plays the role of the formulation of a scientific law, and as it is based on a generalization it cannot be definitely verified. Moreover, the fact that the definition of a R_2-equivalence class is an empirical activity implies that this definition can follow a historical evolution. The attribute that a new measurement system measures is interpreted in terms of already known concepts-operations, and such an attribute becomes a new element of a previously defined R_2-equivalence class (more seldom, a reference on *what* the system is measuring could be missing. This was the case of the thermometer that was introduced during the XVII century: two centuries of research were needed to begin to understand what was measured by that instrument. Indeed, Kuhn (0) notes that «many of the first experiments that used the thermometer seem to be investigations of such an instrument rather than investigations with that»).

The connection between the extensional meaning and the operational one for the attributes is then clear: each R_2-equivalence class of operations can be considered the extensional meaning of the corresponding term of attribute. Consequently, since via R_2 even an indirect relation is allowed between members of such a class, the same attribute could be evaluated by means of operations performed in completely different operative contexts. For example, length is an attribute recognized as measurable on scales spanning several orders of magnitude. Through the relation R_2 operations applied on a microscopic and a macroscopic domain can be made equivalent, and, in principle, a single concept of length results applicable to atoms and galaxies.

5. Consequences and conclusions

5.1. Quantities as general or specific concepts

It has been recognized (cf., for example, (0)) that the term "quantity" is used with two different meanings: in a general sense, for example the concept of length, and a specific one, the length of a given thing at a given time (in the past such a distinction was expressed in terms of “magnitudes”, i.e., quantities in general sense, and “quantities”, i.e., quantities in specific sense (cf., for example, (0)).

The concept of attribute discussed here is used in a general meaning whereas the usual representational definitions of measurement (for example «measurement is the process of empirical, objective assignment of numbers to the attributes of objects and events of the real world, in such a way as to describe them» (0)) seem to subsume a specific meaning, and also the quoted **(0)** defines a measurand as a *specific* quantity.

This distinction *seems* analogous (but we will point out the difference) to that presented in Section 4.2 in terms of “attributes” (viz., quantities with general meanings) and “properties” (viz., quantities with specific meanings). According to the representational point of view and following such a phraseology, the same value is assigned to two different things whenever such things “show the same property”. Therefore, the specific concept of quantity would be expressed as:

IF thing_1 *is* property_1 THEN property_value_1 is assigned
IF thing_2 *is* property_2 THEN property_value_2 is assigned

when distinct things “show distinct properties” and then with distinct values, and:

IF thing_1 *is*
property THEN (the same) property_value is assigned
IF thing_2 *is*

when instead distinct things “show the same property”.

On this basis, a measurement is represented by the IF-THEN construct of the schemes: it is thought of as a mapping from properties / quantities / attributes (the terminology varies in different definitions) to values. The same interpretation was given by B.Russell in terms of “magnitudes”: «measurement demands some one-one relation between the numbers and magnitudes, a relation which may be direct or indirect, important or trivial, according to circumstances» (0).

This view is made even more explicit when bearing in mind that such a mapping is indeed a morphism since «measurement has something to do with assigning numbers that correspond to, or represent, or “preserve” certain observed relations» (0). Therefore a measurement would provide a mapping:

({properties},{relations of {properties}}) $\rightarrow$ ({property values},{relations of {property values}})

so whenever the properties of two things are recognized as related (for example, they show the same property: the relation of sameness holds between such properties), then the corresponding values should be correspondingly related. However, we do not think this is an adequate modelization for the concept of measurement.

Measurement requires an empirical interaction between a thing (and not a property, or a quantity, or an attribute) and a measurement system. It therefore seems reasonable to model such an interaction as a mapping *from things* to symbols. Moreover, if the knowledge of the properties under measurement must precede the measurement itself then the task of measurement would be trivial, at least from an epistemic point of view. Instead of being an activity of “gathering knowledge”, as it is commonly deemed to be, it seems to be reduced to a symbolic assignment, a mere labeling for something that is substantially already known. But why distinguish between properties and property values? We believe that this interpretation calls for the application of Occam's razor: «*entia non sunt multiplicanda sine necessitate*». Do not introduce entities that are not strictly required, and particularly when they are only conceptual ones such as these "properties".

In a model we consider more adequate, a measurement maps things to symbols, and *an attribute is the name for the mapping*:

thing attribute attribute_value (=property)

In the functional notation, this leads to expressions such as:

length(this_table)=1,23 m

where 1,23 m is the value assigned to this_table with respect to the attribute "length". Such expressions are precisely what has been previously called a description.

In this view the concept of attribute in the general sense plays a fundamental role, while any related specific concept seems to be simply useless.

5.2. Measurements as homomorphisms or isomorphisms?

The representational point of view asserts that measurement is formalized as a morphic mapping but does not make clear whether it is an isomorphism or an homomorphism.

It should be recognized that the actual problem is about the injectivity of such a mapping since its surjectivity is not relevant, its range being symbolic and therefore at least partially conventional (indeed, any mapping f from X to Y can be made surjective by simply restricting the range to its image so that $Y=f(X)$). Recalling that an injective homomorphism is called a monomorphism (so that an isomorphism is a surjective monomorphism), in more precise terms, the question becomes: must measurement be formalized as an homomorphism or a monomorphism?

The issue has been already discussed, for example, by Narens (0) who considered that «the choice of homomorphisms as the basis for the representational theory of measurement has never been adequately justified». He declared to prefer «to change the character of the representational theory a little and consider a scale to be an isomorphism between the empirical or qualitative situation and some mathematical situation. The primary reason for this is that isomorphisms preserve truth whereas homomorphisms do not».

The doubt between homomorphisms and monomorphisms seems to be understandable only in view of the ambiguity that remains within the representational point of view on what should be the domain of the morphism that formalizes a measurement. Indeed, the injectivity of the morphism clearly depends on the actual choice of the domain itself.

In the previous Section this point was already taken into account in terms of quantities as general or specific concepts. In view of that discussion, *if* quantities are assumed as specific concepts and measurement is thought of as a mapping from quantities to symbols then the morphism always maps different elements to different values. Such a morphism would then be injective and therefore a monomorphism.

On the other hand, we have tried to justify a general concept of quantity as more adequate to model measurement. In that case, the morphism maps things to symbols and it is apparent that different things can be associated with the same value: hence, in general, the morphism is not injective, and therefore only a homomorphism.

5.3. A proposal for a definition of measurement

Although only indirectly, the previous considerations have some implications on the concept of measurement itself, so we are now able to propose a definition of measurement that partially corrects the representational one.

Let us consider again the definition quoted by (0): «measurement is the process of empirical, objective assignment of numbers to the attributes of objects and events of the real world, in such a way as to describe them».
Instead, we prefer: «measurement is the process of empirical, objective assignment of symbols to things with respect to attributes, in such a way as to describe such things and their relations».

Acknowledgments

My views on these topics have been deeply influenced by the many discussions I have had with Prof. M. Cunietti of Politecnico di Milano (Milan Polytechnic). A draft version of this work was reviewed by Prof. L. Finkelstein of City University of London who gave me some important suggestions to clarify the ideas presented here. To both go my deepest thanks. Finally, I wish to thank a referee for his helpful and encouraging comments.

References

(0) BIPM, ISO, et al., *International Vocabulary of basic and general Terms in Metrology*, II ed., International Organization of Standardization, Geneva, 1993.

(0) D. Krantz, R. Luce, P. Suppes, A. Tversky, *Foundations of Measurement,* Academic Press, New York, Vol.1 (1971), Vol.2 (1989), Vol.3 (1990).

(0) R. Carnap, *Philosophical foundations of Physics*, Basic books, New York, 1966.

(0) J. Piaget, *Psychologie et épistemologie*, Gouthier, Paris, 1970.

(0) S. Stevens, Measurement, Psychophysics, and Utility, in: C. West Churchman and P. Ratoosh (eds.), *Measurement: Definitions and Theories*, J.Wiley, New York, 1959.

(0) F. Roberts, *Measurement Theory*, Addison–Wesley, Reading (Ma), 1979.

(0) C. Singer *et al.*, *History of Technology*, Oxford University Press, Oxford, 1954.

(0) P. Bridgman, *The Logic of modern Physics*, Macmillan, London, 1927.

(0) R. Carnap, *The Logical syntax of Language*, Routledge & Kegan, London, 1937.

(0) E. Nagel, *The Structure of Science*, Hancourr, Brace & World, New York, 1961.

(0) B. Ellis, *Basic Concepts of Measurement*, Cambridge Univ. Press, Cambridge, 1968.

(0) J. Van Brakel, Norms and Facts in Measurement, *Measurement* **2**(2), 45-51 (1984).

(0) L. Carroll, *Alice in Wonderland*, 1865.

(0) C. Hempel, *Philosophy of natural sciences*, Prentice-Hall, Englewood Cliffs (NJ), 1966.

(0) E.A. Abbott, *Flatland - A Romance in many Dimensions*, Seeley, 1884.

(0) M. Schlick, Meaning and Verification, *Philosophical review* **45**(4), 339-369 (1936).

(0) K. Lambert, G. Brittan, *An Introduction to the Philosophy of Science*, Ridgeview, Reseda, California, 1979.

(0) P. Bridgman, *The Way the Things are*, Harvard Univ. Press, Harvard, 1959.

(0) R. Rosen, *Fundamentals of Measurement and Representation of Natural Systems*, North–Holland, Amsterdam, 1978.

(0) J. Borges, *Ficciones*, Sur, Buenos Aires, 1955.

(0) A. Pap, Are physical magnitudes operationally definable?, in: C. West Churchman, P. Ratoosh (eds.), *Measurement: Definitions and Theories*, Wiley, New York, 1959.

(0) J. Barrow, *The World within the World*, Clarendon, Oxford, 1988.

(0) M.L. Dalla Chiara, G. Toraldo di Francia, *Le teorie fisiche*, Boringhieri, Torino, 1981.

(0) M. Cunietti, A. Giussani, The Problem of true Value and its Definition, in: *Proc. Symp. on Measurement Theory & Measurement Error analysis*, IMEKO, 1975.

(0) L. Gonella, Operative Approach to the Theory of Measurement, in: L. Gonella (ed.). *Proc. Workshop on fundamental logical Concepts of Measurement*, IMEKO, 1983.

(0) T. Kuhn, The Function of Measurement in the modern Physical Sciences, *Isis* **52**, 161-190 (1961).

(0) B. Russell, *The principles of mathematics*, Cambridge Univ. Press, Cambridge, 1903.

(0) L. Finkelstein, Measurement: fundamental Principles, in: L. Finkelstein and K. Grattan (eds.), *Concise Encyclopedia of Measurement and Instrumentation*, Pergamon, Oxford, 1994.

(0) L. Narens, *Abstract Measurement Theory*, MIT Press, Cambridge (Ma), 1985.

Symbols

u	the thing under measurement
A	the measured attribute with respect to a given thing u
v	a value for A, belonging to a given set SYMB
Dom(A)	the domain of the attribute A, i.e., the set of things to which A can be applied
op	an operation of evaluation of a thing u with respect to an attribute A
Dom(*op*)	the domain of the operation *op*, i.e., the set of things to which the operation can be applied
$A(u)=v$, or $A_{op}(u)=v$, or $A_{op}(u,t)=v$	a description (the variable t represents the time)
$P(x)$	a unary predicate, formalizing a property
$\underline{P}$	a functor, formalizing a set $P_1,\ldots,P_n$ of exhaustive and mutually exclusive properties. i.e., an attribute
R_1,R_2	relations between operations *op*

$\forall$	universal quantifier ($\forall x$ is read "for all xs …")
$\exists$	existential quantifier ($\exists x$ is read "for at least one x …")
$\cap$	intersection between sets
$\in$	membership relation between elements and sets
$\neq$	not equal relation
$\varnothing$	empty set

Notes

[i] An example of such ambiguity is when one says that “a standard reference *produces* a quantity”. Moreover, things and their attributes are often not distinguished from one another, typically when (almost) always the same attribute is evaluated on the given thing. This is the case, e.g., of the electrical charge or current, that in our terminology are “things”, being the quantity of charge and the intensity of current attributes evaluated on them. On the other hand, it is usually said “a charge of x coulomb” and “a current of y ampère”. This would be like saying, “a table of z meters” instead of “a table whose length is z meters”.

[ii] The analysis presented on the relation between “to be <property>” and not-“to be <property>” can be repeated analogously in this case, for “to have <attribute>” and not-“to have <attribute>”, i.e., to be or not to be in the domain of <attribute>. One can then assert that “this table has a color” but also that not -“this atom has a color”, corresponding to the more usual “this atom has not a color”. In this sense, “color” and “non–color” seem to be the two values that a thing can assume with respect to an attribute at a conceptually higher level than “color” itself. Such an attribute can be called a “meta–attribute” and its values “meta–properties” of the thing. The transition from attributes to meta–attributes is then characterized by a widening of the domain: the meta–attribute having as values “color”, and “non–color” has a wider domain than the attribute “color”: precisely because an atom does not have a color, the attribute “color” is not applicable to it (and a color cannot be indicated with the question “what is its color?”), whereas, for the same reason, the mentioned meta-attribute has value “non–color” for any atom.
Iteratively one could consider meta–meta–attributes, meta–meta–meta–attributes and so on, each time extending the domain of applicability: why is this not the case, and is iteration usually stopped at its first step? Possibly is it a psychological defense against a potential, operatively impossible to manage, *regressio ad infinitum*?

[iii] The time-dependence of the attribute meanings arises the problem when it is still acceptable to maintain the *same* name to denote a shifting concept, and when, on the other hand, the change has been so radical that the new concept requires a new designation. The complexity of such a problem is attested by amount of related literature, mainly in the field of the Philosophy of Language.