Interpreting the basic evaluation equation as an ontological stance

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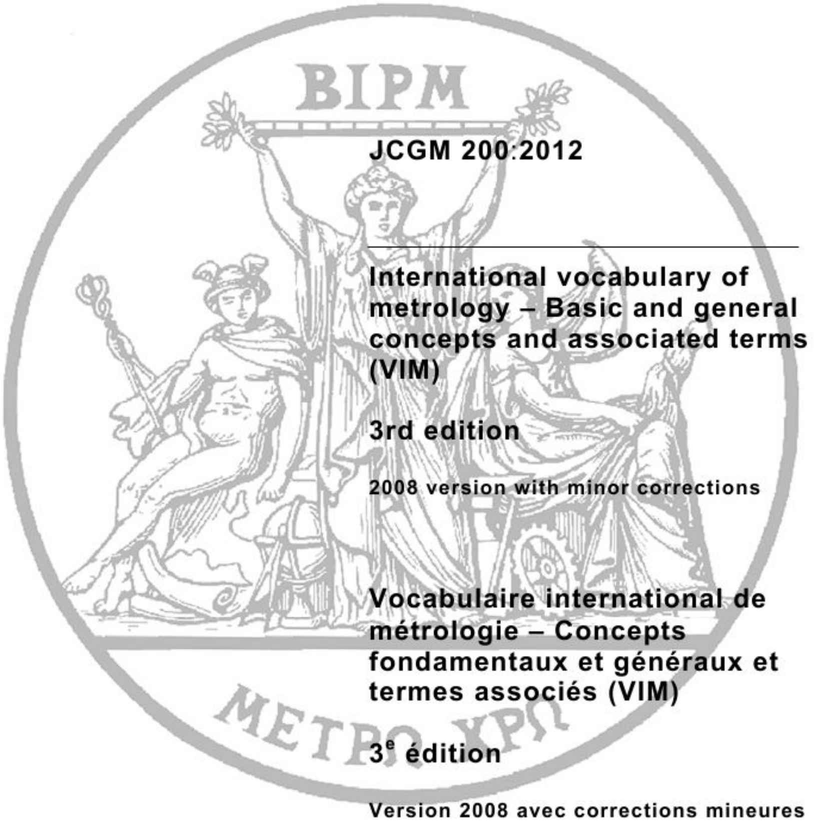
Background: the International Vocabulary of Metrology


BIPM

JCGM 200:2012

International vocabulary of metrology – Basic and general concepts and associated terms (VIM)

3rd edition

2008 version with minor corrections

Vocabulaire international de métrologie – Concepts fondamentaux et généraux et termes associés (VIM)

3e édition

Version 2008 avec corrections mineures

METPΩ XPΩ

https://www.bipm.org/en/committees/jc/jcgm/publications

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International Vocabulary of Metrology

Fourth edition – Second Committee Draft (VIM4 2CD)

31 July 2023

The contents of this document shall not be quoted in any publication

https://www.bipm.org/documents/20126/115700832/VIM4_2CD_clean/c6d0dfb2-ddbf-059e-1f74-9b025c9c59d8

Reporting measurement information

A statement reporting a measurement result [*] is usually presented as [**]

measurand = *measured value*

for example

“the length of object a is $1.234\ \mathrm{m}$”

more formally written

(α) $$\ell(a) = 1.234\ \mathrm{m}$$

What kind of information does (α) convey?

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[*] Some information on measurement uncertainty should be also present, but is omitted here

[**] Terms from the *International Vocabulary of Metrology* (VIM)

Significance of the question

Establishing the kind of information conveyed by a relation like

(α) $$\ell(a) = 1.234\ \mathrm{m}$$

requires us to better understand:

- what are entities like the metre, m (“measurement units”)
- what are entities like $1.234\ \mathrm{m}$ (“values of quantities”)
- what are entities like $\ell(a)$ (“measurands”, “individual quantities”)
- what are entities like ℓ (“general quantities”, “kinds of quantities”)
- if and how (α) depends on models, idealizations, etc
- …

Significance of the question /2

Establishing the kind of information conveyed by a relation like

(α) $$\ell(a) = 1.234\ \mathrm{m}$$

elicits our ontological position about properties, and then quantities:

- from **anti-realism** (properties do not exist; hint: “representationalism”)
- to **realism** (properties do exist; hint: “equationalism”)

Syntactic background

A relation like

(α) $$\ell(a) = 1.234\ \mathrm{m}$$

is syntactically of the form

variable = value

This does not set constraints
on the semantics and the ontology of (α)

Pragmatic background

A relation like

(α) $$\ell(a) = 1.234 \text{ m}$$

may report information other than a measurement result,
either a **declarative** statement (like a prediction or an opinion)
or a **non-declarative** statement (like a specification or a desire)

I focus here on declarative statements,
so that in principle (α) is about states of the world / is either true or false

Don't ask metrologists…

In the context of metrology this issue is usually taken for granted, though there is some confusion about the meaning of

(α) $$\ell(a) = 1.234 \text{ m}$$

For example, in one of the few papers devoted to the subject [*] the "=" in (α) is said to mean "is expressed, modeled, or represented by"

[*] Price, G. (2001). On the communication of measurement results. Measurement, 29, 293-305

Don’t ask metrologists… /2

“**A physical quantity**
is expressed as
the product of a numerical value and a unit:
    physical quantity = numerical value × unit”

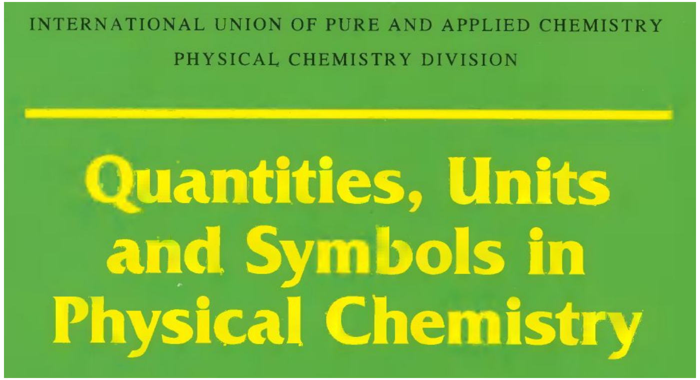
(IUPAP “Red Book”, 1987 revision)

INTERNATIONAL UNION OF PURE AND APPLIED PHYSICS
Commission C2 - SUNAMCO

SYMBOLS, UNITS,
NOMENCLATURE AND
FUNDAMENTAL CONSTANTS
IN PHYSICS

“**The value of a physical quantity**
can be expressed as
the product of a numerical value and a unit:
    physical quantity = numerical value × unit”

(IUPAC “Green Book” 2nd ed, 1993)



Don’t ask metrologists… /3

… also because sometimes someone takes Maxwell’s premise [*]:

> 1.] EVERY expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity.

as it were the definition of what a quantity is…

(*) Maxwell, J.C. (1873). A treatise on electricity and magnetism. Oxford University Press

A step back

The statements reporting measurement results are sometimes presented, e.g., as

“the length in metres of object a is 1.234”

and written

(β) $$\ell_{\mathrm{m}}(a) = 1.234$$

Is there any **fundamental** difference between

(α) $\ell(a) = 1.234\ \mathrm{m}$ and (β) $\ell_{\mathrm{m}}(a) = 1.234$

?

The broader picture

With some differences, this applies also to non-quantitative or non-physical cases like

(α) the blood group
of person z
is A in the ABO System

$$b(z) = \mathrm{A\ in\ ABO\_Sys}$$

(β) the blood group in the ABO System
of person z
is A

$$b_{\mathrm{ABO\_Sys}}(z) = \mathrm{A}$$

Hence, **our subject is neither type-specific nor domain-specific**

(I will work out examples about physical ratio cases mainly because they are more usual)

Standpoint / Hypothesis

While the information conveyed by

(α) $\ell(a) = 1.234\ \mathrm{m}$ and (β) $\ell_{\mathrm{m}}(a) = 1.234$

is operationally the same,

their underlying ontologies can be significantly different

(hint: while (β) is about numbers, (α) is about *something else*)

Let us explore the issue…

Exploring (the simpler) (β)

(β) $\ell_{\mathrm{m}}(a) = 1.234$

$$\begin{tikzcd} D_\ell \arrow[rr, "\ell_{\mathrm{m}}"] \arrow[dr, dashed] & & \mathbb{R}^+ \\ & D_\ell^{\approx} \arrow[ur, dashed] & \end{tikzcd}$$

The function ℓ_{m}, length-in-metres, is a **scale**,
that associates objects-having-length with numbers

$\ell_{\mathrm{m}}(a)$ is the length-in-metres of a, that is a number.

Any two objects a and b in D_ℓ are comparable
by length-related equivalence, and

if $a \approx_\ell b$ then $\ell_{\mathrm{m}}(a) = \ell_{\mathrm{m}}(b)$

ℓ_{m} induces a partition on D_ℓ

if $a \approx_\ell b$ then they have (or: must be mapped to) the same length-in-metres

Any given length can be extensionally identified with an $\approx_\ell$-equivalence class of objects

Exploring (the simpler) (β) /2

Any two objects a and b in D_L are comparable by ℓ-related order and ratio, where

if $a \leq_{\ell} b$ then $\ell_{\mathrm{m}}(a) \leq \ell_{\mathrm{m}}(b)$

if $a +_{\ell} b \approx_{\ell} c$ then $\ell_{\mathrm{m}}(a) + \ell_{\mathrm{m}}(b) = \ell_{\mathrm{m}}(c)$

There can be other scales defined on the same domain D_{ℓ} , e.g., ℓ_{ft},
and such scales can be transformed to each other, i.e., there exists a constant $k_{\mathrm{m}\to\mathrm{ft}}$ such that

for any a in D_{ℓ} , $\ell_{\mathrm{ft}}(a) = k_{\mathrm{m}\to\mathrm{ft}}\, \ell_{\mathrm{m}}(a)$

$$\begin{tikzcd}
 & \mathbb{R}^{+} \arrow[d, "k_{\mathrm{m}\to\mathrm{ft}}"] \\
D_{\ell} \arrow[ur, "\ell_{\mathrm{m}}"] \arrow[r, "\ell_{\mathrm{ft}}"'] & \mathbb{R}^{+}
\end{tikzcd}$$

Length can be extensionally identified with the set $\{\ell_{\mathrm{x}}\}$,
where each scale ℓ_{x} can be obtained from any other ℓ_{y}
by applying a transformation $k_{\mathrm{y}\to\mathrm{x}}$

Consistency or truth

Conditionals like

if $a \approx_{\ell} b$ then $\ell_{\mathrm{m}}(a) = \ell_{\mathrm{m}}(b)$

if $a \leq_{\ell} b$ then $\ell_{\mathrm{m}}(a) \leq \ell_{\mathrm{m}}(b)$

if $a +_{\ell} b \approx_{\ell} c$ then $\ell_{\mathrm{m}}(a) + \ell_{\mathrm{m}}(b) = \ell_{\mathrm{m}}(c)$

are **consistency**, not truth, conditions

This framework may get rid of true values

Summary

Any given length ⇢ an equivalence class of objects

can be extensionally identified with

Length ⇢ a set of transformable functions from objects to numbers

This strategy does not require any explicit involvement of properties / quantities
(the metre is the parameter of a scale)
and emphasizes the **representational** nature of measurement

(β) $\ell_m(a) = 1.234$ can be interpreted as: the object a is ℓ_m-represented by the number 1.234

Measurement

By **Ernest Nagel** (New-York)

The occasion and conditions for measurement. Measurement has been defined as the correlation with numbers of entities which are not numbers

Erkenntnis, 2, 1931, pp. 313-335

Beyond (strict) representationalism?

But is it really the case that
– definitions like: velocity is the first derivative of position
– physical laws like Hooke's law ($F = k\,x$)
– designs and models of measuring instruments and of measurements
– …
involve equivalence classes of objects or sets of transformable functions?

Observation:

(β) $\ell_{\mathrm{m}}(a) = 1.234$ is compatible not only with strict representationalism:

let us explore another position

Beyond representationalism: from (β) to (α)

According to a century-long tradition, relation (β) $\ell_{\mathrm{m}}(a) = 1.234$

may be interpreted as $\ell(a) / \mathrm{m} = 1.234$

and therefore as (α) $\ell(a) = 1.234\ \mathrm{m}$

For example:

"For a physical quantity symbolized by a, [the] relationship is represented in the form

$$a = \{a\} \cdot [a],$$

where $\{a\}$ stands for the numerical value of a and $[a]$ stands for the unit of a."

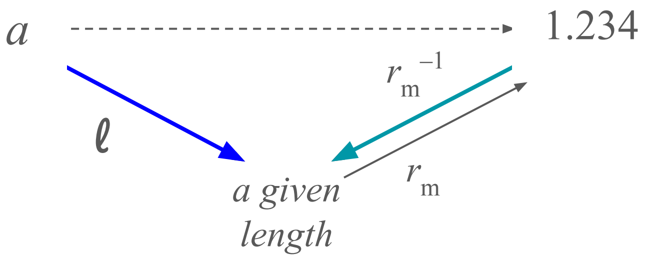
(IUPAP "Red Book", 1987 revision)

(α) is a specific case of (β)

Exploring (the more complex) (α)

(α) $\ell(a) = 1.234\ \mathrm{m}$

The function ℓ, length, associates objects having length with lengths, so that $\ell(a)$ is the length of a

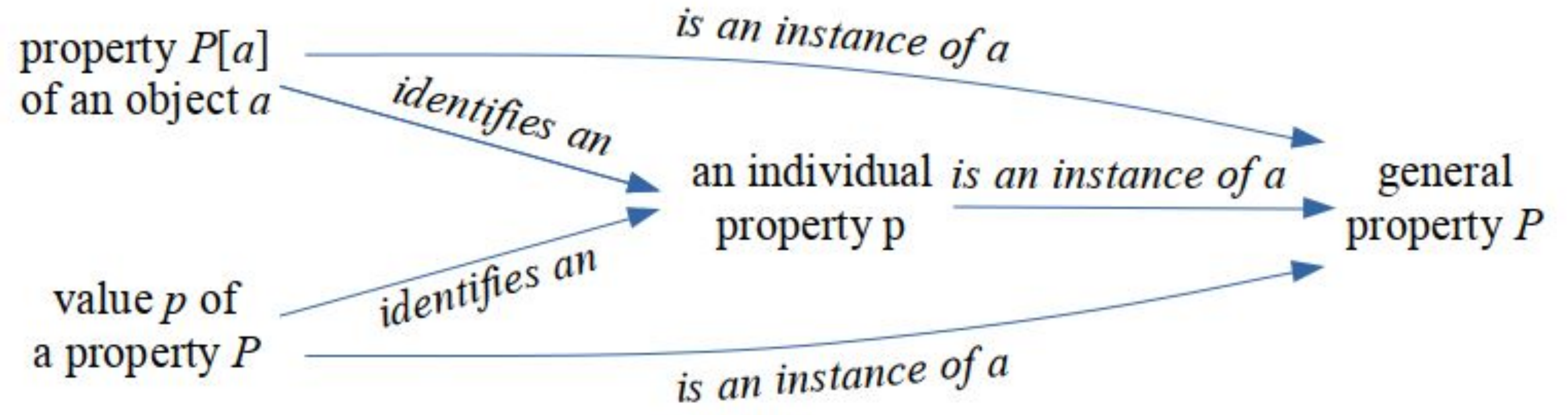


The function r_{m}, in-metres, is a **scale**, that associates lengths (i.e., elements of L) with numbers

(α) means that there is a length that can be presented both as the length of a and as 1.234 times the length identified as the metre:

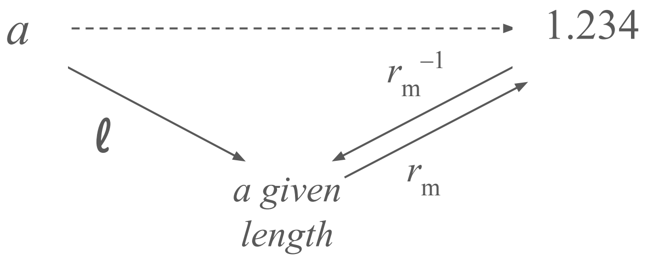
(α) is true if and only if $\ell(a)$ **and** $r_{\mathrm{m}}^{-1}(1.234)$ **are the same length**

About the ontology underlying (α)



Truth, and not only consistency

(α) $\ell(a) = 1.234\ \mathrm{m}$



If (α) is true then the value in it can be called the “**true value**” of the measurand
(as in the definition proposed in the VIM4 2CD:
“value of a quantity of a given object such that the equation relating the quantity and the value is true”)

This framework recovers the role of true values

Some pros of (α)

It provides

- a simple answer to the question: what is a measurement unit?
  (what is the metre? a length)

- an account of comparison of objects with respect to properties in terms of comparison of the properties themselves
  (b is longer than a if the length of b is greater than the length of a,
  and this is an empirical fact independent of numbers, units, scales, etc)

- a simple answer to the question: what is a value of a quantity?
  (what is $1.234\ \mathrm{m}$? a multiple of a unit, and therefore a length)

This explanatory power is obtained thanks to a rich ontology, that explicitly includes properties

Summary /2

(β) $\ell_{\mathrm{m}}(a) = 1.234$

"representationalism"

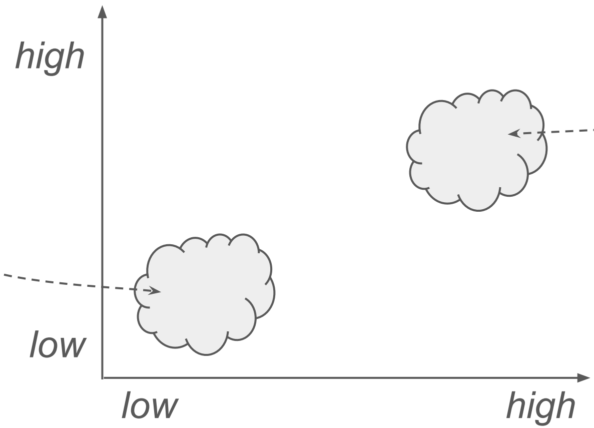
(β) interpreted as such and *nothing else*
→ emphasis on (β) as a **representation**

simpler ontology
more complex interpretations

"equationalism"

(β) interpreted as (α)
→ emphasis on (α) as an **equation**

more complex ontology
simpler interpretations



Moving forward (beyond naive realism)

The analysis so far applies also to properties of abstract / mathematical objects

Let us specify it to empirical properties (measurement is about empirical properties, isn’t it?)

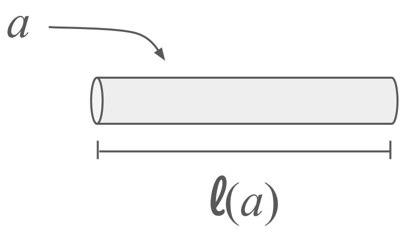
We assume that measurement requires the application of a measuring instrument,
and acknowledge that the property with which the instrument interacts (the “**effective** property”)
and the property that we intend to measure (the “**intended** property”) could be different

The distinction between effective properties and intended properties

- is crucial to understand measurement
- is much more effectively dealt with in the equational framework

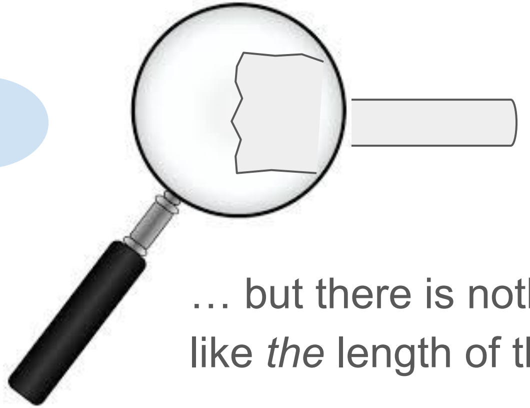
An example

1



We have to measure the length of rod a…

2



… but there is nothing in the world like *the* length of the rod

Then? Depending on our purposes, we have at least two options:

3a we change the measurand (e.g., the distance between two points etc)

3b we **model** the rod as having a length (e.g., as a cylinder)

Exploring (the more complex) (α) /2

If we model the rod a as having a length, we interpret (α) $\ell(a) = 1.234\ \mathrm{m}$ as follows:

$$\begin{tikzcd}
D_\ell \arrow[r, "mod"] \arrow[dr, dashed, "\ell"'] & M_\ell \arrow[r, dashed, "\ell^*_{\mathrm{m}}"] \arrow[d, "\ell^*"'] & \mathbb{R}^+ \arrow[dl, shift right, "r_{\mathrm{m}}^{-1}"'] \\
 & L \arrow[ur, shift right, "r_{\mathrm{m}}"'] &
\end{tikzcd}$$

the rod a is modeled with respect to length:

$$a \rightarrow mod(a)$$

the model of the rod a has a length:

$$mod(a) \rightarrow \ell^*(mod(a))$$

this length is mapped to a number via a scale:

$$\ell^*(mod(a)) \rightarrow r_m(\ell^*(mod(a)))$$

so that, by measurement, we discover that $\ell(a)$ and $1.234\ \mathrm{m}$ are *indistinguishable* and then, in view of our purposes, *identifiable*: $\ell(a) = 1.234\ \mathrm{m}$

Exploring (the more complex) (α) /3

We report $\ell(a) = 1.234\ \mathrm{m}$, not $\ell^*(mod(a)) = 1.234\ \mathrm{m}$

This is based on the acknowledgment that the difference between

$$\begin{tikzcd}
D_\ell \arrow[r, "mod"] \arrow[dr, dashed, "\ell"'] & M_\ell \arrow[r, dashed, "\ell_{\mathrm{m}}^*"] \arrow[d, "\ell^*"'] & \mathbb{R}^+ \arrow[dl, shift right, "r_{\mathrm{m}}^{-1}"'] \\
 & L \arrow[ur, shift right, "r_{\mathrm{m}}"'] &
\end{tikzcd}$$

the length-in-the-world $\ell(a)$

and

the length-in-the-model $\ell^*(mod(a))$

as accounted for by **definitional uncertainty** (*)

is negligible for the current purposes,

as stated by **target uncertainty** (**)

(something like: $|\ell(a) - \ell^*(mod(a))| \approx def\_unc < targ\_unc$)

(*) "... uncertainty resulting from the finite amount of detail in the definition of a measurand", according to the VIM

(**) "... uncertainty specified as an upper limit and decided on the basis of the intended use of measurement results", again according to the VIM

Back to (strict) representationalism?

(α) $\ell(a) = 1.234\ \mathrm{m}$

$$
\begin{tikzcd}
D_\ell \arrow[r, "mod"] \arrow[dr, dashed, "\ell"'] & M_\ell \arrow[r, dashed, "\ell_{\mathrm{m}}^*"] \arrow[d, "\ell^*"] & \mathbb{R}^+ \arrow[dl, shift right, "r_{\mathrm{m}}^{-1}"'] \\
& L \arrow[ur, shift right, "r_{\mathrm{m}}"'] &
\end{tikzcd}
$$

(β) $\ell_{\mathrm{m}}(a) = 1.234$

$$
\begin{tikzcd}
D_\ell \arrow[rr, "\ell_{\mathrm{m}}"] \arrow[dr, dashed] & & \mathbb{R}^+ \\
& D_\ell^{\approx} \arrow[ur, dashed] &
\end{tikzcd}
$$

D_ℓ is expected to be an empirical system, but the existence of a morphism is proved for M_ℓ
(empirical relations apply to empirical objects, such as rods, not to ideal objects, such as cylinders)

The black box approach and the lack of explicit role for properties make it harder to model

- the connection between empirical and ideal entities
- the identification of what is measured
- the role of both definitional and measurement uncertainty

Questions…

(α) $\ell(a) = 1.234\ \mathrm{m}$

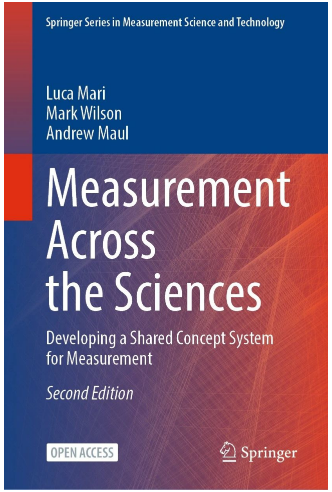
$$
\begin{tikzcd}
D_\ell \arrow[r, "mod"] \arrow[dr, dashed, "\ell"'] & M_\ell \arrow[r, dashed, "\ell_{\mathrm{m}}^{*}"] \arrow[d, "\ell^{*}"'] & \mathbb{R}^{+} \arrow[dl, shift right, "r_{\mathrm{m}}^{-1}"'] \\
 & L \arrow[ur, shift right, "r_{\mathrm{m}}"'] &
\end{tikzcd}
$$

(β) $\ell_{\mathrm{m}}(a) = 1.234$

$$
\begin{tikzcd}
D_\ell \arrow[rr, "\ell_{\mathrm{m}}"] \arrow[dr, dashed] & & \mathbb{R}^{+} \\
 & D_\ell^{\approx} \arrow[ur, dashed] &
\end{tikzcd}
$$

Is perhaps the ontologically richer equational model safer for physical properties, and less justified for psychosocial properties, that are “constructed”?

But even if so, isn’t (strict) representationalism a refusal to open the box and accept the challenge to investigate what is inside?



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Thank you for your attention

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